Compact Finite Difference Schemes For One-Dimensional Helmholtz Equation

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Abstract:

Compact finite difference schemes up to order eight for solving the inhomogeneous Helmholtz equation in one-dimension with Dirichlet and/or Neumann boundary conditions, are developed in this paper. The schemes are implemented to solve a problem with a proper solution. Numerical procedures have been conducted to demonstrate the efficiency of these schemes.

Key Words:

Finite difference method, high order compact finite difference schemes, one dimensional Helmholtz equation.

1. Introduction:

There has been a growing interest in developing higher order accurate discretization methods. In order to obtain high accurate numerical solutions, in the standard finite difference methods for solving ordinary and partial differential equations [6,10], one has either to increase the number of nodes making smaller mesh sizes which require more computing time and storage space or to use high order schemes which require the increase of the stencil of grid points, hence increasing the bandwidth of the stiffness matrix, which makes a fast direct solver difficult. Therefore, compact finite difference schemes are desired to solve differential equation numerically. There are efforts to compute more accurate solutions using limited grid sizes through developing high-order compact finite difference schemes. There are two main approaches for the construction of compact difference schemes: Pade approximation method [2] and Taylor series method [10]. Compact difference schemes are high-order implicit methods which feature higher-order accuracy with smaller stencils and have been used widely in the large area of computational problems, for example, for the steady convection-diffusion problem [11,15], the Poisson equation [1,9,13] and the Helmholtz equations [7,8, 12].

The inhomogeneous one-dimensional Helmholtz equation:

$$u'' + k^2 u = f(x), \quad x \in \Omega \tag{1}$$

where k is a constant and $\Omega = \{x : a < x < b\}$, has been the subject of many investigators. High Order Standard Finite Difference Schemes for equation (1) has been considered by [4]. A noticeable work for the homogeneous equation with Sommerfeld's radiation condition has been done in [5,14]. A sixth-order accurate compact finite difference method for the inhomogeneous equation is given in [7].

This paper attempt to develop up to eighth-order accurate compact finite difference schemes for equation (1) with the Dirichlet boundary condition applied at one end of the interval and the Neumann boundary condition on the other end. To be explicit, the boundary conditions are takon as

$$u'(a) = \alpha \, , \; u(b) = \beta \, . \qquad (2)$$

The methods to be used here are similar to that used in [7].

2 COMPACT FINITE DIFFERENCE SCHEMES :

A uniform grid of the interval $[a,b]$ is used with N uniform segments, so that the grid spacing is $h = \Delta x = \dfrac{b-a}{N}$ and the mesh points are $a = x_1 < x_2 < ... < x_N < x_{N+1} = b$, where $x_{i+1} = x_1 + ih \, , i = 1,2,...,N$. Let $u_i = u(x_i)$ denote the solution of problem (1)-(2) at $x = x_i$ and $u_i^{(n)} = u^{(n)}(x_i)$ denote its nth derivative at $x = x_i$. We shall also let f_i and $f_i^{(n)}$ denote f and its nth derivative at x_i, that is $f_i = f(x_i)$ and $f_i^{(n)} = f^{(n)}(x_i)$.

In order to find an appropriate description of the compact schemes, a Taylor series expansion is performed for the descritized field u_i.

$$u_{i+1} = u_i + hu_i' + \frac{h^2}{2!}u_i'' + \frac{h^3}{3!}u_i^{(3)} + \frac{h^4}{4!}u_i^{(4)} + \frac{h^5}{5!}u_i^{(5)} + \frac{h^6}{6!}u_i^{(6)} + \frac{h^7}{7!}u_i^{(7)} + \frac{h^8}{8!}u_i^{(8)} + \frac{h^9}{9!}u_i^{(9)} + O(h^{(10)}) \qquad (3)$$

$$u_{i-1} = u_i - hu_i' + \frac{h^2}{2!}u_i'' - \frac{h^3}{3!}u_i^{(3)} + \frac{h^4}{4!}u_i^{(4)} - \frac{h^5}{5!}u_i^{(5)} + \frac{h^6}{6!}u_i^{(6)} - \frac{h^7}{7!}u_i^{(7)} + \frac{h^8}{8!}u_i^{(8)} - \frac{h^9}{9!}u_i^{(9)} + O(h^{(10)}) \qquad (4)$$

From equations (3) and (4), we obtain the second order central difference ($\delta_c^1 u_i$) of the first derivative of u_i and the standard second-order central difference $\delta_c^2 u_i$ of the second derivative of u_i as:

$$\delta_c^1 u_i = \frac{u_{i+1} - u_{i-1}}{2h} = u_i' + O(h^2) \qquad (5)$$

$$\delta_c^2 u_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = u_i'' + O(h^2) \qquad (6)$$

Using equations (3)-(6), we have

$$\delta_c^1 u_i = \frac{u_{i+1} - u_{i-1}}{2h} = u_i' + \frac{h^2}{6}u_i^{(3)} + \frac{h^4}{120}u_i^{(5)} + \frac{h^6}{5040}u_i^{(7)} + O(h^{(8)}) \qquad (7)$$

$$\delta_c^2 u_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = u_i'' + \frac{h^2}{12}u_i^{(4)} + \frac{h^4}{360}u_i^{(6)} + \frac{h^6}{20160}u_i^{(8)} + O(h^{(8)}) \qquad (8)$$

2.1 The Eighth-Order Compact Finite Difference Scheme

To obtain an eighth-order compact finite difference scheme for equation (1), we apply δ_c^2 to $u_i^{(6)}$, using (6)

$$u_i^{(8)} = \delta_c^2 u_i^{(6)} + O(h^2)\,. \tag{9}$$

Substituting equation (9) into equation (8), we have

$$\delta_c^2 u_i = u_i'' + \frac{h^2}{12}u_i^{(4)} + \frac{h^4}{360}u_i^{(6)} + \frac{h^6}{20160}\delta_c^2 u_i^{(6)} + O(h^{(8)}) \tag{10}$$

Writing equation (1) in the descritized form, we have

$$u_i'' = -k^2 u_i + f_i \tag{11}$$

From equation (11) , we get

$$u_i^{(4)} = -k^2 u_i'' + f_i'' \tag{12}$$

$$u_i^{(6)} = -k^2 u_i^{(4)} + f_i^{(4)} \tag{13}$$

We substitute equations (11)-(13) into equation (10). The result, after some algebraic manipulation, is

$$\begin{aligned}\delta_c^2 u_i &= \left(1 - \frac{k^2h^2}{12} + \frac{k^4h^4}{360} + \frac{k^4h^6}{20160}\delta_c^2\right)u_i'' \\ &+ \frac{h^2}{12}\left(1 - \frac{k^2h^2}{30} - \frac{k^2h^4}{1680}\delta_c^2\right)f_i'' - \frac{h^4}{360}\left(1 + \frac{h^2}{56}\delta_c^2\right)f_i^{(4)}\end{aligned} \tag{14}$$

An implicit approximation for u_i'' with eighth-order accuracy is given as

$$u_i'' = \frac{\delta_c^2 u_i - \frac{h^2}{12}\left(1 - \frac{k^2h^2}{30} - \frac{k^2h^4}{1680}\delta_c^2\right)f_i'' - \frac{h^4}{360}\left(1 + \frac{h^2}{56}\delta_c^2\right)f_i^{(4)}}{1 - \frac{k^2h^2}{12} + \frac{k^4h^4}{360} + \frac{k^4h^6}{20160}\delta_c^2} \tag{15}$$

Let U_i denote the eighth-order approximation of u_i, that is $u_i = U_i + O(h^8)$.

Substituting for u_i'' from equation (15) into equation (11) using equation (8) together with

$\delta_c^2 f_i'' = \dfrac{f_{i+1}'' - 2f_i'' + f_{i-1}''}{h^2}$ and $\delta_c^2 f_i^{(4)} = \dfrac{f_{i+1}^{(4)} - 2f_i^{(4)} + f_{i-1}^{(4)}}{h^2}$, we obtain after some algebraic manipulation, the following 3-point eighth-order compact finite difference scheme for the one-dimensional Helmholtz equation

$$\begin{aligned} a_{81}U_{i-1} + a_{80}U_i + a_{81}U_{i+1} &= b_{81}f_{i-1} + b_{80}f_i + b_{81}f_{i+1} \\ &+ c_{81}f_{i-1}'' + c_{80}f_i'' + c_{81}f_{i+1}'' + d_{81}f_{i-1}^{(4)} + d_{80}f_i^{(4)} + d_{81}f_{i+1}^{(4)} \end{aligned} \qquad (16)$$

where

$$a_{80} = -2 + k^2h^2 - \frac{k^4h^4}{12} + \frac{3k^6h^6}{1120} \;,\quad a_{81} = 1 + \frac{k^6h^6}{20160},\quad b_{80} = h^2\left(1 - \frac{k^2h^2}{12} + \frac{3k^4h^4}{1120}\right),$$

$$b_{81} = \frac{k^4h^6}{20160},\; c_{80} = \frac{h^4}{12}\left(1 - \frac{9k^2h^2}{280}\right),\; c_{81} = -\frac{k^2h^6}{20160},\; d_{80} = \frac{3h^6}{1120},\; d_{81} = \frac{h^6}{20160}.$$

Invoking the Dirichlet boundary condition $u(b) = \beta$ is straight forward. For the Neumann boundary condition $u'(a) = \alpha$, we need to conduct the eighth-order approximation. Applying δ_c^2 to $u_i^{(5)}$, using (6), we get

$$u_i^{(7)} = \delta_c^2 u_i^{(5)} + O(h^2)\,. \qquad (18)$$

Substituting equation (18) into equation (7), we have

$$\delta_c^1 u_i = u_i' + \frac{h^2}{6}u_i^{(3)} + \frac{h^4}{120}u_i^{(5)} + \frac{h^6}{5040}\delta_c^2 u_i^{(5)} + O(h^{(8)}) \qquad (19)$$

Differentiating equation (11), we get

$$u_i^{(3)} = -k^2 u_i' + f_i' \tag{20}$$

$$u_i^{(5)} = -k^2 u_i^{(3)} + f_i^{(3)} \tag{21}$$

Using $\delta_c^2 u_i' = u_i''' + O(h^2)$, and substitute equations (11), and (20)-(21) into equation (19), we obtain

$$\delta_c^1 u_i = \left(1 - \frac{k^2h^2}{6} + \frac{k^4h^4}{120} - \frac{k^6h^6}{5040}\right)u_i' + \frac{h^2}{6}\left(1 - \frac{k^2h^2}{20} + \frac{k^4h^4}{840} - \frac{k^2h^4}{840}\delta_c^2\right)f_i' + \frac{h^4}{120}\left(1 + \frac{h^2}{46}\delta_c^2\right)f_i^{(3)} + O(h^8) \tag{22}$$

Using $\delta_c^2 f_i' = \dfrac{f_{i+1}' - 2f_i' + f_{i-1}'}{h^2}$, $\delta_c^2 f_i^{(3)} = \dfrac{f_{i+1}^{(3)} - 2f_i^{(3)} + f_{i-1}^{(3)}}{h^2}$ and (5), we obtain after some algebraic simplifications

$$u_{i+1} - u_{i-1} = 2h\alpha\left(1 - \frac{k^2h^2}{6} + \frac{k^4h^4}{120} - \frac{k^6h^6}{5040}\right) - \frac{k^2h^5}{2520}\left(f_{i-1}' + f_{i+1}'\right) + \frac{h^3}{3}\left(1 - \frac{k^2h^2}{21} + \frac{k^4h^4}{840}\right)f_i' + \frac{h^5}{2520}\left(f_{i-1}^{(3)} + f_{i+1}^{(3)}\right) + \frac{h^5}{3}f_i^{(3)} + O(h^8)$$

Replacing u_i by U_i , we see that for the first node ($x = x_1 = a$),

$$U_2 - U_0 = 2h\alpha\left(1 - \frac{k^2h^2}{6} + \frac{k^4h^4}{120} - \frac{k^6h^6}{5040}\right) - \frac{k^2h^5}{2520}\left(f_0' + f_2'\right) + \frac{h^3}{3}\left(1 - \frac{k^2h^2}{21} + \frac{k^4h^4}{840}\right)f_1' + \frac{h^5}{2520}\left(f_0^{(3)} + f_2^{(3)}\right) + \frac{h^5}{3}f_1^{(3)} \tag{23}$$

If we put $i=1$ in equation (16), we have

$$a_{81}U_0+a_{80}U_1+a_{81}U_2=b_{81}f_0+b_{80}f_1+b_{81}f_2 +c_{81}f_0''+c_{80}f_1''+c_{81}f_2''+d_{81}f_0^{(4)}+d_{80}f_1^{(4)}+d_{81}f_2^{(4)} \qquad (24)$$

It is to be noted that if f is not defined outside the interval $[a,b]$, then it must be continued to the left of $x=a$ so that $f_0=f(a-h)$ can be evaluated.

Multiplying equation (23) by a_{81} and add it equation (24), we get

$$\begin{aligned} a_{80}U_1+2a_{81}U_2 &= 2h\alpha a_{81}\left(1-\frac{k^2h^2}{6}+\frac{k^4h^4}{120}-\frac{k^6h^6}{5040}\right)-\frac{a_{81}k^2h^5}{2520}\left(f_0'+f_2'\right) \\ &+\frac{h^3}{3}\left(1-\frac{k^2h^2}{21}+\frac{k^4h^4}{840}\right)f_1'+\frac{h^5}{2520}\left(f_0^{(3)}+f_2^{(3)}\right) \\ &+\frac{h^5}{63}f_1^{(3)}+b_{81}f_0+b_{80}f_1+b_{81}f_2+c_{81}f_0''+c_{80}f_1''+c_{81}f_2'' \\ &+d_{81}f_0^{(4)}+d_{80}f_1^{(4)}+d_{81}f_2^{(4)} \end{aligned} \qquad (25)$$

Equation (25) together with equations (16) for $i=2,\ldots,N$ form a linear system of N equations in N unknowns U_i for $i=1,2,\ldots,N$.

2.2 The sixth-order and fourth-order compact finite difference schemes

The sixth-order and the fourth-order compact finite difference schemes for equation (1) and the sixth-order and the fourth-order approximation of the Neumann boundary condition (2), may be found in a similar way as in the case of the eighth-order [7].

Sixth-order:

$$a_{61}U_{i-1}+a_{60}U_i+a_{61}U_{i+1}=b_{61}f_{i-1}+b_{60}f_i+b_{61}f_{i+1}+c_{61}f_{i-1}''+c_{60}f_i''+c_{61}f_{i+1}'' \qquad (26)$$

$$a_{60}U_1 + 2a_{61}U_2 = 2h\alpha a_{61}\left(1 - \frac{k^2h^2}{6} - \frac{k^4h^4}{120}\right) + b_{61}(f_0 + f_2) + b_{60}f_1 + \frac{a_{61}h^3}{60}(f_0' + f_2')$$
$$+ \frac{a_{61}h^3}{30}\left(9 - \frac{k^2h^2}{2}\right)f_1' + c_{61}(f_0'' + f_2'') + c_{60}f_1'' \quad (27)$$

where

$$a_{60} = -2 + k^2h^2\left(1 - \frac{7k^2h^2}{90}\right), \quad a_{61} = 1 - \frac{k^4h^4}{360},$$

$$b_{60} = h^2\left(1 - \frac{7k^2h^2}{90}\right), \; b_{61} = -\frac{k^2h^4}{360}, \; c_{60} = \frac{7h^4}{90}, \; c_{61} = \frac{h^4}{360}.$$

Fourth-order: $U_{i-1} + a_{40}U_i + U_{i+1} = b_{40}f_i + c_{40}f_i''$ (28)

$$a_{40}U_1 + 2U_2 = 2h\alpha\left(1 - \frac{k^2h^2}{6}\right) + b_{40}f_1 + \frac{h^3}{3}f_1' + c_{40}f_1'' \quad \text{,where} \quad (29)$$

$$a_{40} = -2 + k^2h^2\left(1 - \frac{k^2h^2}{12}\right), \quad b_{40} = h^2\left(1 - \frac{k^2h^2}{12}\right), \quad c_{40} = \frac{h^4}{12}.$$

3. TEST EXAMPLE AND NUMERICAL RESULTS:

An example with a known exact solution is chosen in order to show the performance of the high order compact schemes developed in section 2 using computer programs that implement these schemes. Testing is conducted on the unit interval $[0,1]$ with a uniform mesh size h, and boundary conditions (2) are prescribed on ends of the unit interval. The computations were performed in a MATLAB environment using version 7.6 and was executed on Pentium(R) at 1.86 GHz, RAM 1 GB. The computed solutions and the exact solution are compared with the use of the ℓ^2-norm of the error vector, which is defined for $e=(e_1,e_2,\dots,e_M)$ as $\|e\|_2=\sqrt{\sum_{i=1}^{M}|e_i|^2}$.

Test problem :

$$u''+k^2u=x^2+e^x,\quad 0<x<1$$

$$u(0)=0,\quad u(1)=0$$

with the exact solution

$$u(x)=A\cos kx+B\sin kx+(1/k\,\hat{}\,2)x^2-(2/k^4)+(1/(k^2+1))e^x\ ,$$

where

$$A=(2/k^4)-(1/(k^2+1))\ ,$$

$$B=-A\cos k\,/\sin k+(-1/k^2+2/k^4-e/(k^2+1))/\sin k\ .$$

The eigenvalues of the corresponding completely homogeneous problem are [3]: $(n+1/2)^2\pi^2,\ n=0,1,2,3,\dots$. If k^2 is equal to one of these eigenvalues, then the problem has no solution. We investigate the numerical solutions when k is close to one of the eigenvalues.

The ℓ^2 error norms of the numerical solutions to this problem of $O(h^4)$, $O(h^6)$ and $O(h^8)$ for $N=16,32,64,128$ and for $k=1,10,20,30.40,50,60$ are shown in Table 1. Fig. 1 compares these schemes to the exact solution at $k=1$ and $N=16$. In general there is a significant improvement in the accuracy of $O(h^6)$ over $O(h^4)$ and that of $O(h^8)$ over $O(h^6)$ and in each of these schemes the error norm ℓ^2 decreases with the increase of the number of nodes. However if $k = 1$, $O(h^8)$ decreases with the increase of N and $O(h^6)$ also decrease with the increase of N. This is shown in Fig. 1 and Fig. 2.

When k is large, the accuracy of all schemes in general is poor. However these schemes still provide reasonable approximations of the solution for moderate values of k ($k \leq 60$) despite the high oscillatory property of the solution in this case, Fig. 4.

In Table 2, the ℓ^2-norm of the errors for $O(h^4)$, $O(h^6)$ and $O(h^8)$ schemes for $N = 16, 32, 64, 128$ are compared for $1.5 \leq k < \pi/2$. The purpose of this comparison is to see the behavior of the approximate solutions when k^2 is close to the eigenvalue $\pi^2/4$. As k approaches $\pi/2$, all schemes become sensitive to the value of k and their accuracy is very poor. However, a reasonable accuracy in $O(h^4)$, $O(h^6)$, $O(h^8)$ is still attained as long as k remains respectively within 0.0163, 0.0001266 and 0.0001277 away from the eigenvalue $\pi/2$, Fig. 4.

Table 1. ℓ^2error norm for $O(h^4)$, $O(h^6)$ and $O(h^8)$ compact finite difference schemes for $N = 16, 32, 64, 128$ and for $k = 1, 10, 20, 30.40, 50, 60$.

k	N	$O(h^4)$	$O(h^6)$	$O(h^8)$
1	16	8.6610e-008	1.3721e-010	1.8355e-014
	32	7.5554e-009	3.2364e-012	3.8360e-013
	64	6.6185e-010	1.4166e-012	1.6413e-012
	128	4.9930e-011	2.2475e-011	1.2702e-011
10	16	2.5403e-004	6.3223e-006	4.9343e-008
	32	2.1323e-005	1.3321e-007	2.5985e-010

k	N	$O(h^4)$	$O(h^6)$	$O(h^8)$
	64	1.8559e-006	2.9027e-009	1.4231e-012
	128	1.6322e-007	6.3872e-011	5.2452e-014
20	16	0.0108	0.0013	3.9945e-005
20	32	9.3299e-004	2.3973e-005	1.8430e-007
20	64	7.9103e-005	5.0624e-007	9.7250e-010
	128	7.9103e-005	5.0624e-007	9.7250e-010
30	16	0.0939	0.3005	0.0065
	32	0.0341	0.0015	2.5430e-005
	64	0.0021	2.9467e-005	1.2752e-007
	128	1.7678e-004	6.3482e-007	6.8668e-010
40	16	0.0171	0.0139	0.0031
	32	0.0040	3.1193e-004	9.3387e-006
	64	2.2550e-004	5.6473e-006	4.3483e-008
	128	1.8788e-005	1.1967e-007	2.3039e-010
50	16	0.0372	0.0086	0.0056
	32	0.0043	3.9845e-004	1.7889e-005
	64	1.6085e-004	6.2675e-006	7.5265e-008
	128	1.3059e-005	1.3010e-007	3.9082e-010
60	16	0.0032	0.0032	0.0030
	32	0.8368	0.0010	8.2739e-005
	64	2.8845e-004	1.6917e-005	2.9363e-007
	128	2.3850e-005	3.4207e-007	1.4817e-009

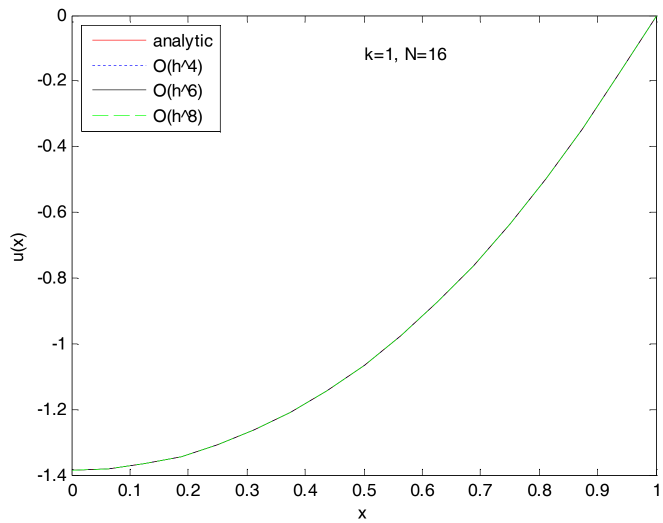

Fig.1. Exact, $O(h^4)$, $O(h^6)$ and $O(h^8)$ solutions at $k=1$ and $N=16$.

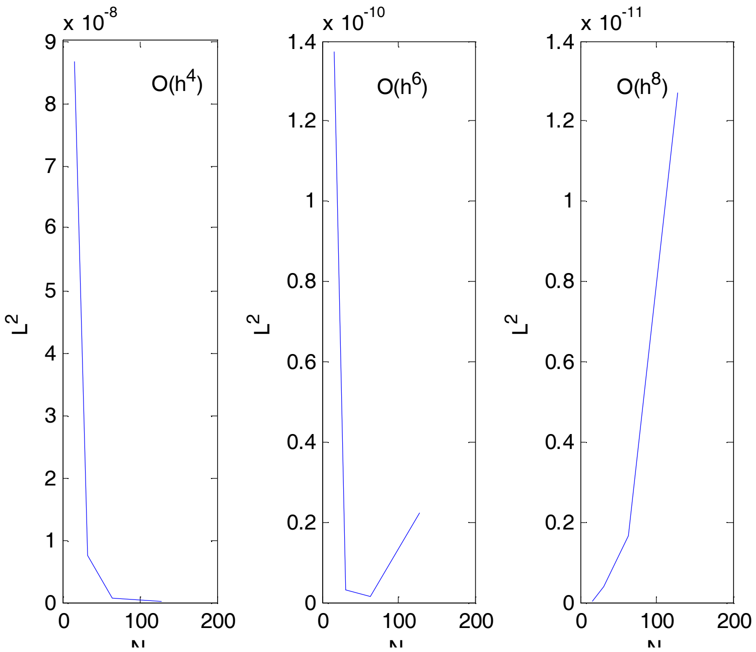

Fig. 2. The effect of the increase of the number of nodes N at fixed wave number ($k = 1$)

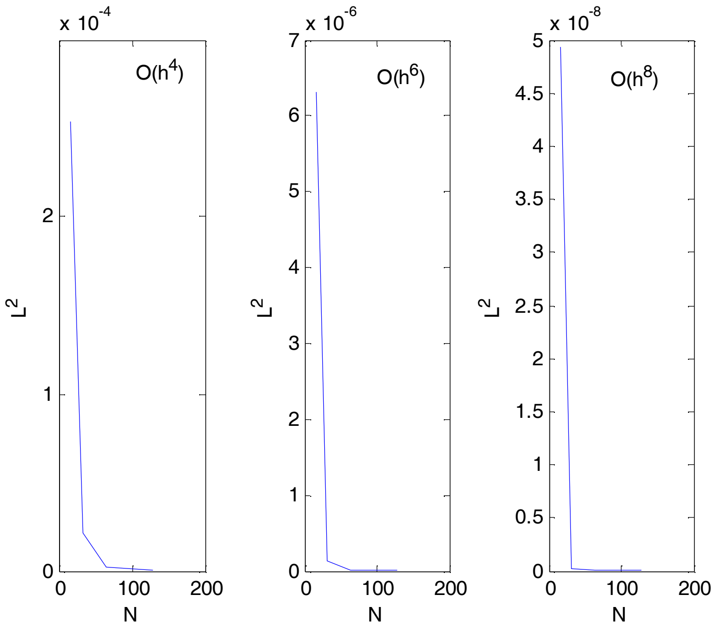

Fig. 3. The effect of the increase of the number of nodes N at fixed wave number ($k = 10$).

Table 2. ℓ^2-norm of the errors for various finite difference schemes in examples 2 for $N = 16, 32, 64, 128$ when k approaches $\pi/2$

k	N	$O(h^4)$	$O(h^6)$	$O(h^8)$
1	16	8.6610e-008	1.3721e-010	1.8355e-014
	32	7.5554e-009	3.2364e-012	3.8360e-013
	64	6.6185e-010	1.4166e-012	1.6413e-012
	128	4.9930e-011	2.2475e-011	1.2702e-011
1.5000000	16	4.0509e-006	7.0325e-009	7.0513e-012
	32	3.5627e-007	1.5469e-010	2.4396e-014
	64	3.1449e-008	3.3103e-012	5.0499e-014
	128	2.7792e-009	5.7106e-013	1.2455e-012
1.5217835	16	1.3831e-004	8.5577e-008	2.3110e-011
	32	1.2032e-005	1.8913e-009	5.5808e-011
	64	1.0552e-006	7.6758e-013	4.8167e-010
	128	9.2379e-008	6.8147e-010	2.7077e-009
1.5381211	16	3.3181e-004	2.0669e-007	1.4891e-011
	32	2.8864e-005	4.4547e-009	1.1008e-010
	64	2.5308e-006	3.5696e-010	1.2168e-009
	128	2.2481e-007	3.6915e-009	5.2885e-011
1.5544587	16	0.0014	8.8574e-007	1.7510e-010
	32	1.2288e-004	1.9623e-008	9.9630e-010
	64	1.0778e-005	2.2108e-009	1.1651e-009
	128	9.5867e-007	1.1031e-008	3.6851e-008
1.57028575	16	1.5335	9.6770e-004	1.0503e-007
	32	0.1334	2.0761e-005	2.0012e-007
	64	0.0117	2.3305e-006	1.0306e-006
	128	0.0010	9.6813e-006	2.5401e-006

k	N	$O(h^4)$	$O(h^6)$	$O(h^8)$
1.5706686625	16	24.5334	0.0155	4.8455e-007
	32	2.1372	3.3670e-004	7.6709e-006
	64	0.1875	2.1267e-005	3.9840e-005
	128	0.0167	2.0105e-004	3.4362e-004
1.57073248125	16	97.9598	0.0620	1.5313e-005
	32	8.5464	0.0013	6.3396e-005
	64	0.7495	7.7310e-005	1.4050e-004
	128	0.0662	2.6923e-004	0.0019

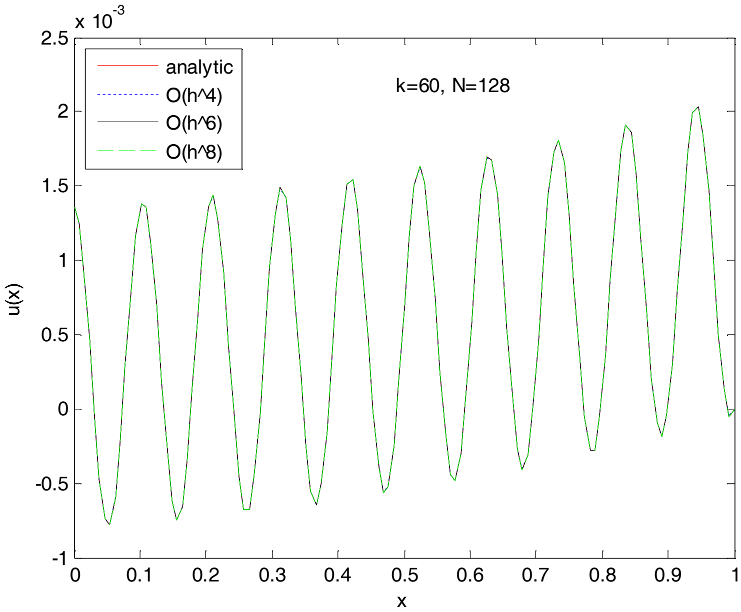

Fig. 4. Exact, $O(h^4)$, $O(h^6)$ and $O(h^8)$ solutions at $k = 60$ and $N = 128$.

4. CONCLUSIONS :

Compact finite difference schemes up to order eight for solving the inhomogeneous Helmholtz equation in one-dimension with Dirichlet and/or Neumann boundary conditions, were developed in this paper.

It is found that a significant improvement in the accuracy of $O(h^6)$ over $O(h^4)$ and that of $O(h^8)$ over $O(h^6)$ and in each of these schemes the error norm ℓ^2 decreases with the increase of the number of nodes. However if $k = 1$, $O(h^8)$ decreases with the increase of N and $O(h^6)$ also decrease with the increase of N. When k is large, the accuracy of all schemes in general is poor but reasonable approximations of the solution for moderate values of k ($k \leq 60$) despite the high oscillatory property of the solution are still available. As k approaches $\pi/2$, all schemes become sensitive to the value of k and their accuracy is very poor. However, a reasonable accuracy in $O(h^4)$, $O(h^6)$, $O(h^8)$ is still attained as long as k remains respectively within 0.0163, 0.0001266 and 0.0001277 away from $\pi/2$.

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