On Propositional Calculus
The Soundness and The Completeness
of The Non-Formal Systems

($\mathscr{L}_{DSi}$, $1 \le i \le 4$)

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Introduction:

The propositional calculus is a branch of mathematic logic some times called propositional logic, it deals with the study of mathematical and logic, it divides into two mains branches.

- Non Axiomatic logical systems (normal logical systems) .
- Axiomatic logical systems (the axiomatic logic).

In the study of non- Axiomatic logical systems we use a natural deduction system without axioms, which has an empty axiom set. to study and proof

Thermos of the deduction systems DSi, 1≤ i ≤ 4

1. Language and definitions:

1-1 Atomic proposition: An atomic proposition is a sentence contains only one content either true or falls. The small letters of the alphabet (a,b,c …etc) standing as atomic proposition.

1-2 Operators: symbols denoting the following connectives (or logical operators): ¬, ∧, ∨, →, ↔.

1-3 Parentheses: Left and right parentheses: (,) , { [(,)] }

1-4 Complex proposition: a complex proposition is a composition of more than one atomic proposition with some operators and parentheses, the capital letters of the alphabet (A , B, C) standing as complex proposition.

1-5 well formed formula (wff): A well formed formula (wff) is a set of complex propositions is recursively defined by the following rules:

- Basis: Letters of the alphabet (usually capitalized such as A, , B, ,C, D , etc.) or the Greek alphabet (χ , φ, ψ)are well-formed formulas wffs is recursively defined by the following rules:

- Inductive clause I: If φ is a wff, then ¬ φ is a wff.
- Inductive clause II: If φ and ψ are wffs, then (φ ∧ ψ), (φ ∨ ψ),

(φ → ψ),and (φ ↔ ψ) are wffs.

1.6 Rules of inferences:

A rule of inference is a valid argument used to deduct a new wff from a previous wff the following are some of rules of inferences:

R_1 : Simplification	p ∧ q ├ p	Simp
R_2 : Com mutative	p ∧ q ├ q ∧ p	Com
R_3: Conj unction	p , q ├ p ∧ q	Conj

1.7 Rules of manipulation **:**

Proposition (1.1**) :** If *A* and *A* → *B* are tautologies, then so is *B* .

Proof. Suppose that *A* and *A* → *B* are tautologies, and that *B* is not. Then

there is an assignment of truth values to the statement letters appearing in *A* or in *B* which gives *B* the value F. But it must give *A* the value T since *A* is a tautology, and so it gives *A* → *B* the value F. This contradicts the assumption that *A* → *B* is a tautology. Hence *B* must be a tautology.

Rules of manipulation and substitution.

1.8 Rules of substitution:

Proposition (1.2): Let A be a wff in which the statement letters

P_1 , P_2 ,........, P_n appear, and let A_1 , A_2 ,......., A_n be any wffs. If A is a tautology then the statement form B , obtained from A by replacing each occurrence of P_i by A_i ($1 \leq i \leq n$) throughout, is a tautology also, i.e. substitution in a tautology yields a tautology.

Proof: Let A be a tautology and let P_1 , P_2 ,........, P_n be the statement letters appearing in A . Let A_1 , A_2 ,......., A_n be any statement forms. Assign any truth values to the statement letters which appear in A_1 , A_2 ,......., A_n The truth value

that *B* now takes is the same as that which *A* would have taken if the values

which A_1 , A_2 ,......., A_n take had been assigned to P_1 , P_2 ,........, P_n respectively, namely T. Hence *B* takes value T under any assignment of truth values, i.e. *B* is a tautology.

Now consider the statement form $((A \wedge A) \rightarrow B)$. $(A \wedge A)$, which appears in this form, is logically equivalent to A (since $((A \wedge A) \equiv A)$is a tautology). If were place $(A \wedge A)$ by A , we get ($A \rightarrow B$). Now($A \rightarrow B$) is logically equivalent to $((A \wedge A) \rightarrow B)$. Again this is an instance of general proposition substitution

1.9 A proof:

We will use a natural deduction system, which has no axioms; or, equivalently, which has an empty axiom set. Derivations using our calculus will be laid out in the form of a list of numbered lines, with a single wff and a justification on each line. Any given wff considered to be assumptions and written in the top of the proof . The conclusion will be on the last line. A derivation will be considered complete if every line follows from previous ones by correct application of a rule.

2.0 A Theorem:

The last wff in the proof called a theorem .

2.0 <u>The deduction system DS1</u>

In this section of this paper discussion and proofs of theorems of the non- formal systems DS1,ds2,DS3,DS4 will be presented.

Rules of inferences of DS1:

1. $(\mathcal{A} \wedge \mathcal{B}) \vdash \mathcal{A}$ *Simplification*
2. $(\mathcal{A} \wedge \mathcal{B} \vdash (\mathcal{B} \wedge \mathcal{A})$ *Commutative*
3. $\mathcal{A} , \mathcal{B} \vdash (\mathcal{A} \wedge \mathcal{B})$ *Conjunction*

Theorem*2- 1-1:* $\mathcal{A} \wedge (\mathcal{B} \supset C) \vdash \mathcal{A}$

Proof

1) $\mathcal{A} \wedge (\mathcal{B} \supset C)$ *assum.*

2) $\mathcal{A}$ *1 , simp.*

$\mathcal{A} \wedge (\mathcal{B} \supset C) \vdash \mathcal{A}$

Theorem 2-1-2:$(\mathcal{B} \vee C) \wedge \mathcal{E} \vdash \mathcal{E}$

1) 1. $(\mathcal{B} \vee C) \wedge \mathcal{E}$ *assumption*

2) 2. $\mathcal{E} \wedge (\mathcal{B} \vee C)$ *1, com.*

3) $\mathcal{E}$ *2, simp.*

$(\mathcal{B} \vee C) \wedge \mathcal{E} \vdash \mathcal{E}$

Theorem 2-1-3: $C \wedge (D \wedge E) \vdash D$

Proof:

1) $C \wedge (D \wedge E)$ assumption

2) $(D \wedge E) \wedge C$ 1, com.

3) $D \wedge E$ 2, simp.

4) 4) D 3, simp.

$C \wedge (D \wedge E) \vdash D$

Theorem 2-1-4 : $A \vee D,\ B \wedge C \vdash\ C \wedge (A \vee D)$

Proof :

1. $A \vee D$ assumption
2. $B \wedge C$ assumption
3. $C \wedge B$ 2, com.
4. C 3, simp.
5. $C \wedge (A \vee D)$ 4,1, conj.

$A \vee D,\ B \wedge C \vdash\ C \wedge D$

Theorem 2-1- 5:$(A \wedge B) \wedge C \vdash B \wedge C$

Proof :

1. $(A \wedge B) \wedge C$ assumption
2. $A \wedge B$ 1, simp.
3. $C \wedge (A \wedge B)$ 1, com.
4. C 3, simp.
5. $B \wedge A$ 2, com.

6. $\mathcal{B}$ 5, simp.

7. $\mathcal{B} \wedge C$ 4, 6, conj.

$(\mathcal{A} \wedge \mathcal{B}) \wedge C \vdash \mathcal{B} \wedge C$

The deduction system DS2

Rules of inferences of DS2

1. $(\mathcal{A} \vee \mathcal{B}), \neg\mathcal{A} \vdash \mathcal{B}$ Disjunctions syllogism(DS)
2. $(\mathcal{A} \vee \mathcal{B}) \vdash (\mathcal{B} \vee \mathcal{A})$ Commutative
3. $\mathcal{A} \vdash (\mathcal{A} \vee \mathcal{B})$ Addition

Theorem 2-2-1:-

$\neg\mathcal{B}, \mathcal{A} \vee \mathcal{B} \vdash \mathcal{A}$

Proof

1. $\neg\mathcal{B}$ assumption
2. $\mathcal{A} \vee \mathcal{B}$ assumption
3. $\mathcal{B} \vee \mathcal{A}$ Com
4. $\mathcal{A}$ 3 , 1 ,DS

$\therefore \neg\mathcal{B}, \mathcal{A} \vee \mathcal{B} \vdash \mathcal{A}$

Theorem 2-2-2:-

$C \wedge \mathcal{D} \vdash \mathcal{D} \vee \mathcal{E}$

Proof

1. $C \wedge \mathcal{D}$ assumption
2. $\mathcal{D} \wedge C$ 1 , Com
3. $\mathcal{D}$ Simp

4. $\mathcal{D} \vee \mathcal{E}$ Conj

$\therefore \mathcal{C} \wedge \mathcal{D} \vdash \mathcal{D} \vee \mathcal{E}$

Theorem 2-2- 3:-

$(\mathcal{A} \vee \mathcal{B}) \wedge \neg\mathcal{B} \vdash \mathcal{A}$

Proof

1. $(\mathcal{A} \vee \mathcal{B}) \wedge \neg\mathcal{B}$ assumption
2. $\mathcal{A} \vee \mathcal{B}$ 1,simp
3. $\neg\mathcal{B} \wedge (\mathcal{A} \vee \mathcal{B})$ 1 , Com
4. $\neg\mathcal{B}$ 3 , Simp
5. $\mathcal{B} \vee \mathcal{A}$ 2 , Com
6. $\mathcal{A}$ 5 , 4 , DS

$\therefore (\mathcal{A} \vee \mathcal{B}) \wedge \neg\mathcal{B} \vdash \mathcal{A}$

Theorem2-2- 4:-

$\neg(\mathcal{A} \vee \mathcal{B}), (\mathcal{C} \supset \mathcal{D}) \vee (\mathcal{A} \vee \mathcal{B}), \neg\mathcal{D} \vdash (\mathcal{C} \supset \mathcal{D}) \wedge (\mathcal{E} \vee \neg\mathcal{D})$

Proof

1. $\neg(\mathcal{A} \vee \mathcal{B})$ assumption
2. $(\mathcal{C} \supset \mathcal{D}) \vee (\mathcal{A} \vee \mathcal{B})$ assumption
3. $\neg\mathcal{D}$ assumption
4. $(\mathcal{A} \vee \mathcal{B}) \vee (\mathcal{C} \vee \mathcal{D})$ 2 , Com
5. $\mathcal{C} \supset \mathcal{D}$ 2 , 1 , DS

6. $\neg D \vee E$ 3 , Add

7. $E \vee \neg D$ 6 , Com

8. $(C \supset D) \wedge (E \vee \neg D)$ 5 , 7 , conj

$\therefore \neg(A \vee B), (C \supset D) \vee (A \vee B), \neg D \vdash (C \supset D) \wedge (E \vee \neg D)$

Theorem 2-2-5:-

$\neg(B \supset C) \wedge A, (E \supset D) \vee (B \supset C) \vdash (D \vee A) \wedge (E \supset D)$

Proof

1. $\neg(B \supset C) \wedge A$ assumption
2. $(E \supset D) \vee (B \supset C)$ assumption
3. $\neg(B \supset C)$ 1 , Simp
4. $A \wedge \neg(B \supset C)$ 1 , Com
5. A 4 , Simp
6. $(B \supset C) \vee (E \supset D)$ 2 , Com
7. $E \supset D$ 6 , 3 , DS
8. $A \vee D$ 5 , Add
9. $D \vee A$ 8 , Com
10. $(D \vee A) \wedge (E \supset D)$ 9 , 7 , Conj

1. The deduction system DS3

Rules of inferences of DS3

1$(A \supset B), A \vdash B$ Modus Ponnens (MP)

2. $(A \supset B), \neg A \vdash \neg A$ Modus Tollens (MT)

Theorem 2-3-1:-

$A \supset B , A \vdash B$

Proof

1. $A \supset B$ assumption
2. A assumption
3. B 1 , 2 , MP

$\therefore A \supset B , A \vdash B$

Theorem 2-3-2:-

$\neg A \supset \neg B , \neg\neg B \vdash \neg\neg A$

Proof

1. $\neg A \supset \neg B$ assumption
2. $\neg\neg B$ assumption
3. $\neg\neg A$ 1 , 2 , MT

$\therefore \neg A \supset \neg B , \neg\neg B \vdash \neg\neg A$

Theorem 2-3-3 :-

$A \wedge (A \supset B) \vdash B$

Proof

1. $A \wedge (A \supset B)$ assumption
2. A 1 , Simp.
3. $(A \supset B) \wedge A$ 1 , Com.
4. $A \supset B$ 3 , Simp
5. B 2 , 4 , MP

$\therefore A \wedge (A \supset B) \vdash B$

Theorem2-3-4:-

$(A \supset B) \wedge (B \supset C), \neg C \vdash \neg A$

Proof

1. $(A \supset B) \wedge (B \supset C)$ assumption
2. $\neg C$ assumption
3. $A \supset B$ 1 , Simp
4. $(B \supset C) \wedge (A \supset B)$ 1 , Com
5. $B \supset C$ 4 , Simp
6. $\neg B$ 2 , 5 , MT
7. $\neg A$ 3 , 6 , MT

$\therefore (A \supset B) \wedge (B \supset C), \neg C \vdash \neg A$

Theorem 2-3-5 :-

$(A \supset B) \wedge (B \supset C), C \supset D, A \vdash D$

Proof

1. $(A \supset B) \wedge (B \supset C)$ assumption
2. $C \supset D$ assumption
3. A assumption

4. $A \supset B$ 1 , Simp

5. $(B \supset C) \wedge (A \supset B)$ 1 , Com

6. $B \supset C$ 5 , Simp

7. B 3 , 4 , MP

8. C 6 , 7 , MP

9. D 2 , 8 , MP

$\therefore (A \supset B) \wedge (B \supset C) , \neg C \vdash \neg A$

The deduction system DS4

Rules of inferences of DS4

1. $(A \supset B), (B \supset C \vdash (A \supset C)$ Hypothetical Syllogism (HS)
2. $(A \supset B), (C \supset D), (A \vee C \vdash (B \vee D)$ Constructive Dilemma(CD)

Theorem 2-4-1:- $\mathcal{A} \supset \mathcal{B} , \mathcal{C} \supset \mathcal{A} \vdash \mathcal{C} \supset \mathcal{B}$

Proof

1) $\mathcal{A} \supset \mathcal{B}$ assumption

2) $\mathcal{C} \supset \mathcal{A}$ assumption

3) $\mathcal{C} \supset \mathcal{B}$ 2 , 1 , HS

$\therefore \mathcal{A} \supset \mathcal{B} , \mathcal{C} \supset \mathcal{A} \vdash \mathcal{C} \supset \mathcal{B}$

Theorem 2-4-2:- $\mathcal{A} \supset \mathcal{B}, \mathcal{A} \vee C, C \supset \mathcal{D} \vdash \mathcal{B} \vee \mathcal{D}$

Proof

1) $\mathcal{A} \supset \mathcal{B}$ assumption

2) $\mathcal{A} \vee C$ assumption

3) $C \supset \mathcal{D}$ assumption

4) $\mathcal{B} \vee \mathcal{D}$ 1 , 2 , 3 , CD

$\therefore \mathcal{A} \supset \mathcal{B}, \mathcal{A} \vee C, C \supset \mathcal{D} \vdash \mathcal{B} \vee \mathcal{D}$

Theorem 2-4-3:- $\mathcal{D} \supset (\mathcal{A} \supset \mathcal{B}), \mathcal{D} \wedge C, C \supset (\mathcal{E} \supset \mathcal{A}) \vdash \mathcal{E} \supset \mathcal{B}$

Proof

1) $\mathcal{D} \supset (\mathcal{A} \supset \mathcal{B})$ *assumption*

2) $\mathcal{D} \wedge C$ *assumption*

3) $C \supset (\mathcal{E} \supset \mathcal{A})$ *assumption*

4) $\mathcal{D}$ *2 , simp*

5) $C \wedge \mathcal{D}$ *2 , com*

6) C *5 , simp*

7) $\mathcal{E} \supset \mathcal{A}$ *3 , 6 , MP*

8) $\mathcal{A} \supset \mathcal{B}$ *1 , 4 , MP*

9) $\mathcal{E} \supset \mathcal{B}$ *7 , 8 , HS*

$\therefore \mathcal{D} \supset (\mathcal{A} \supset \mathcal{B}), \mathcal{D} \wedge C, C \supset (\mathcal{E} \supset \mathcal{A}) \vdash \mathcal{E} \supset \mathcal{B}$

Theorem 2-4-4:- $A \vee B, (B \supset D) \wedge (A \supset D) \vdash \neg(D \vee E) \vee (E \vee D)$

Proof

1) $A \vee B$ *assumption*

2) $(B \supset D) \wedge (A \supset E)$ assumption

3) $B \supset D$ 2 , simp

4) $(A \supset E) \wedge (B \supset D)$ 2 , com

5) $A \supset E$ 4 , simp

6) $E \vee D$ 1 , 3 , 5 , CD

7) $(E \vee D) \vee \neg(D \vee E)$ 6 , add

8) $\neg(D \vee E) \vee (E \vee D)$ 7 , com

$\therefore A \vee B , (B \supset D) \wedge (A \supset D) \vdash \neg(D \vee E) \vee (E \vee D)$

Theorem 2-4-5:- $(\mathcal{A} \supset \mathcal{B}) \wedge \mathcal{C} , \mathcal{D} \supset \mathcal{E} , \mathcal{C} \supset \mathcal{D} \vdash \mathcal{B} \vee \mathcal{E}$

Proof

1) $(\mathcal{A} \supset \mathcal{B}) \wedge \mathcal{C}$ assumption

2) $\mathcal{D} \supset \mathcal{E}$ assumption

3) $\mathcal{C} \supset \mathcal{D}$ assumption

4) $\mathcal{A} \supset \mathcal{B}$ 1 , simp

5) $\mathcal{C} \wedge (\mathcal{A} \supset \mathcal{B})$ 1 , com

6) $\mathcal{C}$ 5 , simp

7) $\mathcal{C} \supset \mathcal{E}$ 2 , 3 , HS

8) $\mathcal{C} \vee \mathcal{A}$ 6 , add

9) $\mathcal{E} \vee \mathcal{B}$ 4 , 7 , 8 , CD

10) $\mathcal{B} \vee \mathcal{E}$ 9 , com

$\therefore (\mathcal{A} \supset \mathcal{B}) \wedge \mathcal{C} , \mathcal{D} \supset \mathcal{E} , \mathcal{C} \supset \mathcal{D} \vdash \mathcal{B} \vee \mathcal{E}$

3- The soundness and completeness of the DSi, 1≤ i ≤ 4

In this part of the paper we will prove the soundness and the completeness of the non-formal systems ($\mathscr{A}$**DSi**) , 1≤ i ≤ 4 .

For both systems DS_i we suggest defining a symbol ($\mathscr{A}$**DSi**) to represent the set of all previous theorems DS_i , in Otherwise:

$\mathscr{A}$**DSi** = { DS_i , 1≤ i ≤ 4 }.

3-1 Definition :(contradiction).

contradiction is a wff that is ⊥ under any possible T assignment of truth values of the wff .

Such propositions are called un-satisfiable. Conversely, a contradiction is ¬T.

3-2 Definition(soundness 1).

If $\mathscr{A}$**DSi** is a set of theorems , and $\boldsymbol{\varphi}$ is a single wff , we say a deductive is sound if

$\mathscr{A}$**DSi** $\vdash \boldsymbol{\varphi} \supset$ $\mathscr{A}$**DSi** $\models \boldsymbol{\varphi}$

to mean that $\boldsymbol{\varphi}$ may be derived from $\mathscr{A}$**DSi** using only the rules of inference.

Remark *.*

Every theorem in DS_i , $1 \leq i \leq 4$, $1 \leq i \leq 4$ **is** T

3-3 Definition a model:

A model is a deductive system consisting a set of finite assumption , and a theorem $\mathscr{A}$**DSi**.

3-4 Definition.

If $\mathscr{A}$**DSi** is consistent in deduction systems and if there is no wff $\boldsymbol{\varphi}$ such that $\mathscr{A}$**DSi** $\vdash \boldsymbol{\varphi}$ and $\mathscr{A}$**DSi** $\vdash \neg\boldsymbol{\varphi}$. Otherwise, $\mathscr{A}$**DSi** is D-inconsistent.

Remark. If $\mathscr{A}$**DSi** is a tautology then ($\neg\mathscr{A}$**DSi**) is not satisfiable.

3-5 Definition.

If $\mathscr{A}$**DSi** is deductive complete if it is deductive consistent and for every formula $\boldsymbol{\varphi}$**,** $\mathscr{A}$**DSi** $\vdash \boldsymbol{\varphi}$ or $\mathscr{A}$**DSi** $\vdash \neg\boldsymbol{\varphi}$.

3 -6 Definition (soundness 2).

If $\mathscr{A}$**DSi** is a set of wffs , and $\boldsymbol{\varphi}$ is a single wff, we say a deductive is sound if $\mathscr{A}$**DSi** is satisfiable then $\mathscr{A}$**DSi** is deduction consistent.

Remark. An argument is sound if and only if :

1. The argument is valid.
2. All of its premises are true.

3-7 Definition (completeness 1).

If $\mathscr{A}$**DSi** is a set of wffs , and $\boldsymbol{\varphi}$ is a single wff , we say a deductive is sound if :

$\mathscr{A}$**DSi** $\models \boldsymbol{\varphi} \supset$ $\mathscr{A}$**DSi** $\vdash \boldsymbol{\varphi}$.

to mean that, for every model $\mathcal{M}$, if $\mathcal{M} \models \mathscr{A}$**DSi** , then $\mathcal{M} \models \boldsymbol{\varphi}$.

3-8 Definition (completeness 2).

If $\mathscr{A}$**DSi** is a set of wffs , and $\boldsymbol{\varphi}$ is a single wff , we say a deductive is sound if $\mathscr{A}$**DSi** is deduction consistent then $\mathscr{A}$**DSi** is satisfiable.

3-9 The Completeness Theorem

An inspection of the set $\mathscr{A}$**DSi** of of formulae shows that every member of $\mathscr{A}$**DSi** is valid. Note that if for wffs $\boldsymbol{\varphi}$ and **ψ**,

if $\models \varphi$ and $\models \varphi \supset \psi$ then $\models \psi$.

3-10 Theorem (soundness)

If $\mathscr{A}$**DSi** $\vdash \boldsymbol{\varphi}$ then $\mathscr{A}$**DSi** $\models \boldsymbol{\varphi}$ **.**

3-11 Theorem (Godel Completeness Theorem)

If $\mathscr{A}$**DSi** $\models \boldsymbol{\varphi}$ then $\mathscr{A}$**DSi** $\vdash \boldsymbol{\varphi}$.

3-12.Proposition .

Theorems 3-11 and 3-12 are equivalent.

Proof.

First, we assume that Theorem 3-11 is true and prove that Theorem 3-12 follows. Then, we assume that Theorem 3-12 is true and prove that Theorem

3-11 follows.

Suppose Theorem 3-11 is true. We want to show that Theorem 3-12 follows.

To that end, suppose that $\mathscr{A}$**DSi** is consistent. We must show that there is a model $\mathcal{M}$ such that $\mathcal{M} \models \mathscr{A}$**DSi**.

$\mathscr{A}$**DSi** is consistent. Thus, for every formula **ψ** , $\mathscr{A}$**DSi** $\nvdash$ (**ψ** $\wedge \neg$**ψ**). Thus, by the contra positive of Theorem 3.10, it follows that $\mathscr{A}$**DSi** $\not\models$

($\psi \wedge \neg\psi$). That is, it is not the case that every model that makes $\mathscr{A}$**DSi** true also makes ($\psi \wedge \neg \psi$) true. Thus, there is a model in which $\mathscr{A}$**DSi** is true and ($\psi \wedge \neg\psi$).

Thus, there is a model in which $\mathscr{A}$**DSi** is true, as required.

Thus, Theorem 3.11 entails Theorem 3.12.

Now, suppose Theorem 3.11 holds. And suppose that $\mathscr{A}$**DSi** $\models \varphi$. Then there is no model of $\mathscr{A}$**DSi**, $\neg\varphi$. Thus, by the contra positive to Theorem 3.12, $\mathscr{A}$**DSi** , $\neg\varphi$ is not consistent. That is,

$\mathscr{A}$**DSi**, $\varphi \vdash (\psi \wedge \neg \psi)$

It follows from this that

$\mathscr{A}$**DSi** $\vdash (\neg\varphi \subset (\psi \wedge \neg \psi)$

Thus ,

$\mathscr{A}$**DSi** $\vdash (\neg (\psi \wedge \neg \psi) \subset \varphi)$

And, since $\mathscr{A}$**DSi** $\vdash \neg (\psi \wedge \neg\psi)$, by modus ponens we have that

$\mathscr{A}$**DSi** $\vdash \phi$

as required. Thus, Theorem 3-12 entails Theorem 3-11.Thus, Theorem 3-11 and Theorem 3-12 are equivalent.

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