Semi-analytic Method With High Order Finite Difference For Laplace's Equation

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Abstract:

The semi-analytic method is used to solve the two dimensional Laplace's equation. Finite difference schemes up to eighth-order are used with respect to one variable. The analytic solution is then obtained for the resulting system of ordinary differential equations in the other variable. The method is implemented to solve a problem with exact solution. Numerical procedures have been conducted to demonstrate the efficiency of the schemes.

Key Words: Semi-analytic method, finite difference method, high order finite difference schemes, two dimensional Laplace's equation.

1. Introduction

Laplace equation is a second order partial differential equation (PDE) that appears in many areas of science and engineering, such as electricity, fluid flow, and steady heat conduction. Finite difference schemes are often used in the numerical solution of PDE's [5,8]. There are many numerical techniques related to the finite difference method that are used for obtaining the solution of Laplace equation such as the successive over-relaxation method, the implicit alternative direction method [1], the method of false transients and the method of lines [6,7]. The method of lines (MOL) is a general technique which has found interesting applications in the numerical treatment of PDE's.. The basic idea of the MOL (which is also called semi-discretization approach) is to replace the derivatives of all but one of the independent variables in the PDE with algebraic approximations using finite difference relationships. Once this is done, the PDE is transformed into a system of ordinary differential equations (ODEs) which can be solved using initial or boundary value problem techniques for ODEs. For the elliptic PDEs, the semi-analytical method of lines consists of using difference approximation for the second-order derivative in one of the spatial directions followed by solving analytically the resulting system of second-order differential equations. Subramanian and White [9] used central difference approximation for the second-order derivative in one of the spatial directions followed by solving analytically the resulting system of second-order differential equations analytically by casting them in first-order form and solving the resulting set of first-order equations by using the matrix exponential. In this paper we use up to eighth-order finite difference approximation for the second-order derivative in one of the spatial directions and then solve analytically the resulting system of second-order differential equations.

2. Semi-Discretization Of Laplace's Equation

We consider the two dimensional Laplace's equation on the rectangle $\Omega = \{(x, y): 0 < x < x_f, 0 < y < y_f\}$, with Dirichlet boundary conditions;

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (x, y) \in \Omega \tag{1}$$

$$u(0, y) = u_0, \qquad 0 < y < y_f \tag{2a}$$

$$u(x_f, y) = u_f, \quad 0 < y < y_f \tag{2b}$$

$$u(x, 0) = g(x), \quad 0 < x < x_f \tag{2c}$$

$$u(x, y_f) = k(x), \; 0 < x < x_f \tag{2d}$$

where u_0 and u_f are constants and g and k are given functions.

We discretize along the x axis using a uniform grid of the interval $[0, x_f]$ with uniform segments, so that the grid spacing is $h = \Delta x = \frac{x_f}{N}$ and the mesh points are $0 = x_1 < x_2 < ... < x_N < x_{N+1} = x_f$, where $x_{i+1} = x_1 + ih, i = 1, 2, ..., N$. We let $u_i = u_i(y) = u(x_i, y)$, $i = 1 : N+1$ so that u_i denotes the solution of problem (1)-(2) at $x = x_i$ and y, $\frac{\partial^2 u_i}{\partial x^2} = \frac{\partial^2 u_i}{\partial x^2}(x_i, y)$ denotes its second derivatives with respect to x at (x_i, y). We shall also let $g_i = g(x_i)$ and $k_i = k(x_i)$.

We shall use the second order ($O(h^2)$), fourth order ($O(h^4)$), sixth order ($O(h^6)$), and eighth order ($O(h^8)$) difference approximations of the second derivative with respect to x at the mesh nodes. For the second, fourth, sixth and eighth order approximations; the internal nodes are respectively $i=2,...,N$, $i=3,...,N-1$, $i=4,...,N-2$ and $i=5,...,N-3$. The rest of the nodes in each case are boundary nodes. The coefficients of the difference approximations of order $2L$ for the P^{th} derivative at the internal nodes may be calculated from :

$$u_i^{(P)}+O(h^{2L})=\frac{1}{h^P}\sum_{k=-L}^{L}C_k u_{i+k} ,$$

and the coefficients of the boundary nodes $\{2 , 3, 4\}$ ($\{$N , N-1, N-2$\}$) may be calculated from :

$$u_i^{(P)}+O(h^{2L})=\frac{1}{h^P}\sum_{k=-L+M}^{L+M}C_k u_{i+k} ,$$ with $M=3,2,1$ ($M=-3,-2,-1$) .

The coefficients of the difference approximations of $O(h^2)$, $O(h^4)$, $O(h^6)$ and $O(h^8)$ for the second derivative at the internal nodes are given in table 1. The coefficients of the difference approximations of $O(h^8)$ for the second derivative at the boundary nodes are given in table 2. For those of $O(h^2)$, $O(h^4)$ and $O(h^6)$, see [2,3].

We now use finite difference approximation of $O(h^8)$ to estimate the right hand side of

$$\frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 u}{\partial x^2} . \quad (3)$$

Using table 1 and table 2, the following system of ODEs in the variable y is obtained.

$$\frac{\partial^2 U}{\partial y^2}=\frac{1}{5040h^2}MU , \quad (4)$$

where $U=[U_1,U_2,...,U_{N-1}]^T=[u_2,u_3,...,u_N]^T$ (T denotes the transpose), the $(N-1)\times(N-1)$ matrix M is constructed from table 1 and table 2. Note that $u_1=u_0$ and $u_{N+1}=u_f$. The matrix M has distinct eigenvalues

e_i, $i = 1,2,...,N-1$ and therefore is similar to a diagonal matrix $D = diag\,(e_1, e_2, ..., e_{N-1})$ [4]. So there exists an invertible matrix P such that $P^{-1}MP = D$ (5)

Let the vector $V = [V_1, V_2, ..., V_{N-1}]^T$ be given by $V = P^{-1}U$ or

$$U = PV \tag{6}$$

Substituting (6) into (4), multiplying by P^{-1} and using (5), we obtain

$$\frac{\partial^2 V}{\partial y^2} - \frac{1}{5040h^2} DV = 0 \,. \tag{7}$$

If we let $\mu_i^2 = \dfrac{e_i}{5040h^2}$, system (7) can be cast as

$$\frac{\partial^2 V_i}{\partial y^2} - \mu_i V_i = 0, \quad i = 1,2,...,N-1 \tag{8}$$

For each i, equation (8) has the general solution

$$V_i(y) = A_i \cosh(\mu_i y) + B_i \sinh(\mu_i y) \tag{9}$$

where A_i and B_i are arbitrary constants.

We set $G = [g_2, g_3, ..., g_N]^T$, $K = [k_2, k_3, ..., k_N]^T$, $A = [A_1, A_2, ..., A_{N-1}]^T$, $B = [B_1, B_2, ..., B_{N-1}]^T$, $P^{-1} = (P_{ij}^*)$, $S_s = [\sinh(\mu_1 y_f), \sinh(\mu_2 y_f), ..., \sinh(\mu_{N-1} y_f)]^T$, $S_{ac} = [A_1 \cosh(\mu_1 y_f), A_2 \cosh(\mu_2 y_f), ..., A_{N-1} \cosh(\mu_{N-1} y_f)]^T$, and $S_{bs} = [B_1 \sinh(\mu_1 y_f), B_2 \sinh(\mu_2 y_f), ..., B_{N-1} \sinh(\mu_{N-1} y_f)]^T$.

Applying the boundary conditions (2c) , we have

$G = U|_{y=0} = PV|_{y=0} = PA$, or $A = P^{-1}G$. So that

$$A_i = \sum_{j=1}^{N-1} P_{ij}^* g_{j+1} \tag{10}$$

Applying the boundary conditions (2d) , we have

$K = U|_{y=y_f} = PV|_{y=y_f} = P\{S_{ac} + S_{bs}\}$, or $S_{bs} = P^{-1}K - S_{ac}$. Or

$$B_i = \frac{1}{\sinh(\mu_i y_f)}\left\{\sum_{j=1}^{N-1} P_{ij}^* K_{j+1} - A_i \cosh(\mu_i y_f)\right\}. \tag{11}$$

With these expressions for A_i and B_i in (10) and (11), $V_i(y)$ in (9) is completely determined. The solution of problem (1)-(2) is then determined from (6).

3. Test Problems and Numerical Results

An example for which the exact solution is known, is now used to test the performance of the method described above for solving problem (1)-(2). The computations were performed in a MATLAB environment using version 7.6 and was executed on Pentium(R) at 1.86 GHz, RAM 1 GB. The computed solutions and the exact solution are compared with the use of the two norms; the ℓ^2-norm and the ℓ^∞-norm of the error vector, defined respectively for $e = (e_1, e_2, ..., e_M)$ as

$\|e\|_2 = \sqrt{\sum_{i=1}^{M} |e_i|^2}$ and $\|e\|_\infty = \underset{1 \le i \le M}{Max} |e_i|$.

Example .

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 1, \quad 0 < y < 1$$

$u(0,y)=0, \quad 0<y<1$

$u(1,y)=0, \quad 0<y<1$

$u(x,0)=0, \quad 0<x<1$

$u(x,1)=k(x)=\sin(\pi x), \quad 0<x<1,$

for which the exact solution is $u(x,y)=\dfrac{\sinh(\pi y)\sin(\pi x)}{\sinh(\pi)}$.

Table 3 provides comparison of exact and numerical solutions for various values of points (x,y) and various values of number of nodes N. Table 4 contains error norms ℓ^2 and ℓ^∞ using schemes of $O(h^2)$, $O(h^4)$, $O(h^6)$ and $O(h^8)$ for $N=16,32,64$. Table 5 Shows the effect of increase of number of nodes on the ℓ^2 error norm for various difference orders. This is depicted in Figure 1. At a fixed number of Nodes, the accuracy of the numerical solution increases with the increase of the order of the finite difference. For a fixed finite difference approximation, the accuracy of the numerical solution increases with the increase of node numbers (decreases of the step h) up to a certain value of N (a certain value of h), after which the accuracy of the numerical solution starts to fluctuate. This happens when the truncation error is of $O(h^8)$ with $h \approx 1/72$ (i.e. $\approx O(1e-015)$) as shown in figure 2.

Table 1. The coefficients of the difference approximations of $O(h^2)$, $O(h^4)$, $O(h^6)$ and $O(h^8)$ for the second derivative at the internal nodes.

coefficients	Order 2	Order 4	Order 6	Order 8
C_{-4}	--	--	--	-1/560
C_{-3}	--	--	1/90	8/315
C_{-2}	--	-1/12	-3/20	-1/5
C_{-1}	1	4/3	3/2	8/5
C_{0}	-2	-5/2	-49/18	-205/72
C_{1}	1	4/3	3/2	8/5
C_{2}	--	-1/12	-3/20	-1/5
C_{3}	--	--	1/90	8/315
C_{4}	--	--	--	-1/560

Table 2. The coefficients of the difference approximations of $O(h^8)$ for the second derivative at the boundary nodes .

	Node 2	Node 3	Node 4	Node N-2	Node N-1	Node N
C_{-4}	--	--	--	--	--	--
C_{-4}	--	--	--	--	--	-29/560
C_{-4}	--	--	--	--	47/5040	599/1260
C_{-4}	--	--	--	-1/560	-3/35	-39/20
C_{-3}	--	--	47/5040	1/70	7/20	47/10
C_{-2}	--	-29/560	-19/140	-7/180	-37/45	-529/72
C_{-1}	363/560	39/35	29/20	-1/20	9/8	153/20
C_{0}	8/315	-331/180	-118/45	11/8	1/5	-83/20
C_{1}	-83/20	1/5	11/8	-118/45	-331/180	8/315
C_{2}	153/20	9/8	-1/20	29/20	39/35	363/560
C_{3}	-529/72	-37/45	-7/180	-19/140	-29/560	--
C_{4}	47/10	7/20	1/70	47/5040	--	--
C_{5}	-39/20	-3/35	-1/560	--	--	--
C_{6}	599/1260	47/5040	--	--	--	--
C_{7}	-29/560	--	--	--	--	--

Table 3. Exact and numerical solutions in examples 1 for various values of points (x, y) with $N = 16, 32, 64$.

N	$x = y =$	Exact	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$
16	0.0625	0.003338243620	0.003349732454	0.003337815328	0.003338257526	0.003338243188
	0.3125	0.082596669301	0.082842437863	0.082595281613	0.082596738910	0.082596667204
	0.5625	0.241329291897	0.241825668546	0.241328005808	0.241329396292	0.241329288904
	0.8125	0.306964259653	0.307245113211	0.306959243728	0.306964415071	0.306964254580
32	0.0313	0.0008345738267	0.000835294630	0.000834567062	0.000834573885	0.000834573826
	0.2813	0.0671429695415	0.067194365732	0.067142956765	0.067142970142	0.067142969536
	0.5313	0.2205270225890	0.220647611234	0.220527057529	0.220527023586	0.220527022582
	0.7813	0.3173184059796	0.317402825737	0.317318325639	0.317318407334	0.317318405968
64	0.0156	0.00020864365858	0.00020868875172	0.0002086435538	0.0002086436588	0.0002086436585
	0.2656	0.05997325663209	0.05998488622296	0.0599732580035	0.0599732566373	0.0599732566320
	0.5156	0.20992479429536	0.20995433082845	0.2099248000641	0.2099247943050	0.2099247942953
	0.7656	0.319572323588884	0.319595057793893	0.31957232520610	0.3195723236008	0.3195723235888

Table 4. Error norms ℓ^2 and ℓ^∞ using differences of $O(h^2)$, $O(h^4)$, $O(h^6)$ and $O(h^8)$ for $N = 16, 32, 64$.

N	Norm	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$
16	ℓ^2	0.0043	4.9595e-005	1.7051e-006	5.2755e-008
	ℓ^∞	0.0057	5.8693e-005	2.3141e-006	7.1028e-008
32	ℓ^2	0.0022	3.2812e-006	3.2608e-008	2.7298e-010
	ℓ^∞	0.0028	4.0766e-006	4.5277e-008	3.6827e-010
64	ℓ^2	0.0011	2.6051e-007	5.8180e-010	2.2423e-012
	ℓ^∞	0.0014	3.6359e-007	8.2934e-010	3.0312e-012

Table 5. The effect of increase of the number of nodes on the ℓ^2 error norm for various difference orders.

N	Norm	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$
16	ℓ^2	0.0043	4.9595e-005	1.7051e-006	5.2755e-008
24	ℓ^2	0.0029	1.0169e-005	1.7173e-007	2.5344e-009
32	ℓ^2	0.0022	3.2812e-006	3.2608e-008	2.7298e-010
40	ℓ^2	0.0017	1.3877e-006	8.9226e-009	4.7256e-011
48	ℓ^2	0.0014	7.0453e-007	3.0911e-009	1.1486e-011
56	ℓ^2	0.0012	4.0802e-007	1.2618e-009	4.0335e-012
64	ℓ^2	0.0011	2.6051e-007	5.8180e-010	2.2423e-012

N	**Norm**	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$
72	ℓ^2	9.6259e-004	1.7880e-007	2.9405e-010	5.4381e-013
80	ℓ^2	8.6633e-004	1.2947e-007	1.6218e-010	3.0485e-012
88	ℓ^2	7.8757e-004	9.7556e-008	9.3879e-011	5.1071e-012
96	ℓ^2	7.2194e-004	7.5775e-008	6.1131e-011	2.9810e-012
104	ℓ^2	6.6640e-004	6.0252e-008	2.7235e-011	5.5936e-012
112	ℓ^2	6.1880e-004	4.8821e-008	2.2381e-011	1.6472e-012
120	ℓ^2	5.7755e-004	4.0184e-008	1.1428e-011	1.8957e-012
128	ℓ^2	5.4145e-004	3.3508e-008	7.9295e-012	1.2086e-011

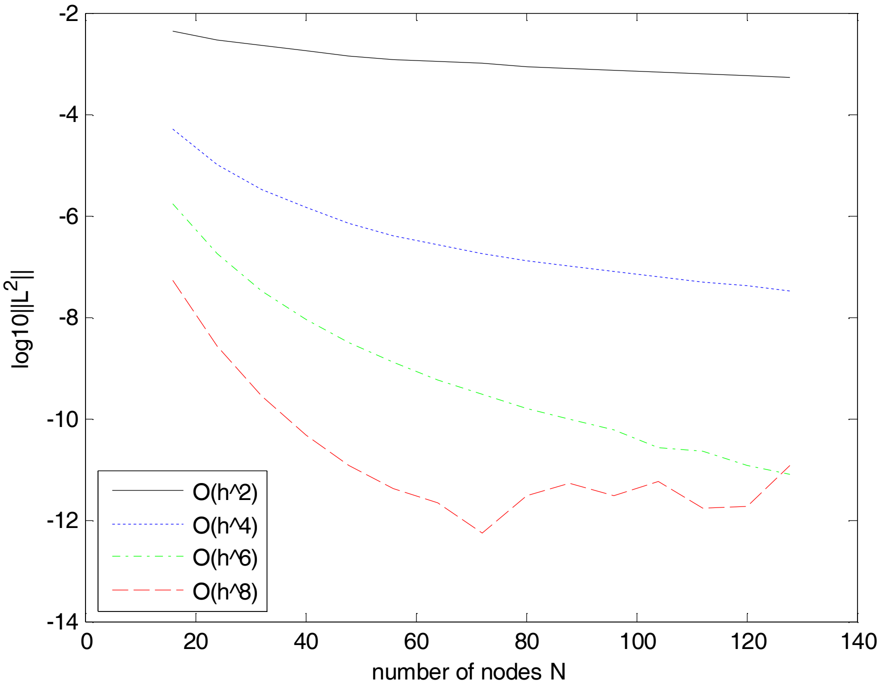

Fig.1 The effect of increase of number of nodes on the ℓ^2 error norm

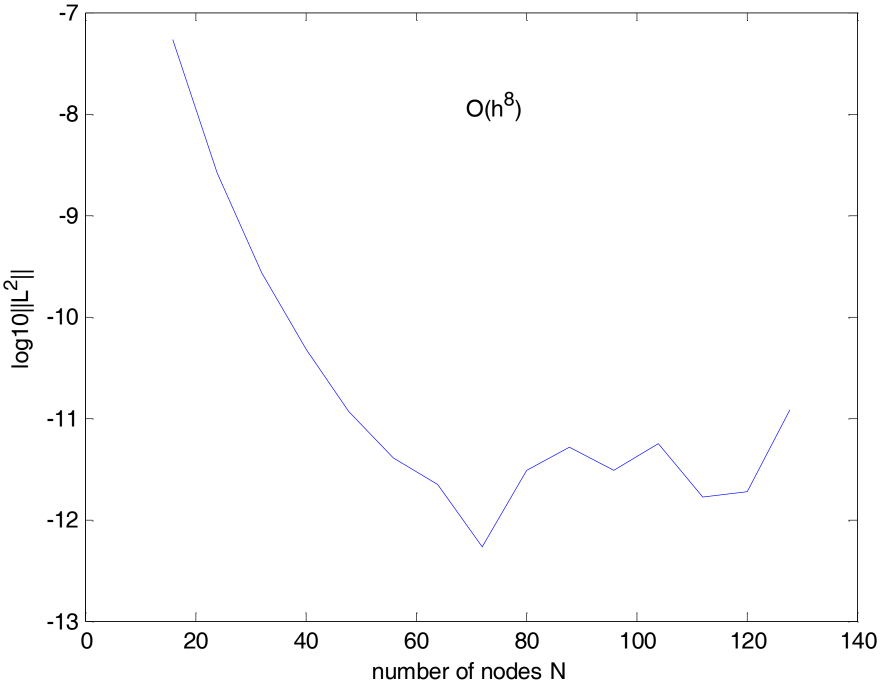

Fig.2 The effect of increase of number of nodes on the ℓ^2 error norm for $O(h^8)$

4. Conclusions

In this paper the two-dimensional Laplace equation is dealt with using the semi-analytic method. Finite difference approximation for the second-order derivative in one of the spatial directions up to eighth-order is used and then we solved analytically the resulting system of second-order differential equations. Comparisons of the results obtained from these schemes with exact solution showed that these schemes provide an efficient high accuracy methods for solving the problem.

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