Polynomial Equations from Ancient to Modern Times – a Review

Hana Almoner Looka
Dept. of Mathematics
Belgrade University

Abstract:

The purpose of this study is to learn more about the methods for solving polynomial equations, and to describe some interesting stories about Arab and Western scholars' efforts to find a general formula to solve cubic, quartic, and quintic equation. In fact this interesting history is unknown to many teachers, students and also university professors in Libya. In addition we explain some topics which are related with polynomial equations, and we give a short overview of modern research related to polynomial equations.

1. Introduction:

There are several reasons why solving cubic and quartic equation was important in the history of mathematics. For instance mathematicians took both complex numbers and negative numbers more seriously because of their importance in solving cubic and quartic equations.

Another important reason is that it has led to a new study, and development of theory of equations, culminating in the 19th century with the proof of unsolvability of the quintic equations by a number of mathematicians.

But very little is known in Libya about this history. Because of that I had chosen this topic; also discussing such topics in class can make students eager to study it and love mathematics with passion.

There is no doubt that this is a very exciting topic; the history of cubic and quartic equations can be an excellent introduction for students to a lot of areas of mathematics such as: algebra, geometric constructions, number theory, numerical analysis, group theory, complex number theory.

Arabic mathematics has been also involved in solving polynomial equations, as we shall see in this paper (namely in cubic and quartic equations).

In the finale of this paper we explain how some constructions are related with polynomial equations. Also, we will give an overview of modern research.

2. Background information and definition:

Before we begin to talk about the subject of the paper it is necessary to review definition of polynomial, cubic equation, quartic equation and quintic equation.

2-1 The general form of algebraic equation is $f(x) = 0$ where

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \ \rightarrow (1),$$

is a polynomial function of degree n , n is a positive integer, a_n , a_{n-1} , ... a_0 are numbers called coefficients.

2-2 Complex numbers ($\mathbb{C}$): A complex number is a number that can be expressed in the form $z = a + ib$ where a and b are real numbers and i is the imaginary unit, satisfying $i^2 = 1$.

- The conjugate of a complex number $a + ib$ is the complex number $a - ib$.

2-3 The Fundamental Theorem of Algebra (FTA): tells us that every polynomial equation of n-th degree has at least one root in the set of complex numbers.

It follows $f(x) = a_n x + a_{n-1} x + \cdots + a_1 x + a_0 = \sum_{i=o}^{n} a_i x^i$, $(a_n \neq 0)$

We can write $f(x)$ as:

$f(x) = a_n (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n) = a_n \prod_{i=1}^{n} (x - \alpha_i)$, where the points α_i are real or complex roots of polynomial.

- Sometimes in mathematics the solutions of any polynomial equation are called zeros.
- Case $n = 3$ (the polynomial is cubic, i.e. equation is of third degree). The general form of cubic equation is $ax^3 + bx^2 + cx + d = 0 \longrightarrow (2)$, where $a \neq 0$, but any or all of b, c and d can be zero, and coefficients b, c and d are real or complex numbers.
- Case $n = 4$ (the polynomial is quartic, i.e. equation is of fourth degree). The general form of quartic equation is $ax^4 + bx^3 + cx^2 + d = 0 \longrightarrow (3)$, where $a \neq 0$.
- If b and $d = 0$ ($ax^4 + cx^2 + e = 0$) this is equation also called biquadratic equation (special case of quartic equation).
- Case $n = 5$ (the polynomial is quintic, i.e. equation is of fifth degree). The general form of quintic equation is

  $$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0 \longrightarrow (4),$$

  where $a \neq 0$.

2-4 Construction in Geometry means to draw shapes, angles or lines accurately using only straightedge and compass.

- A compass is a tool which can be used to draw circle through a given point, or arcs thereof whose radii are sufficiently small, and for measuring lengths.
- A straightedge (unmarked ruler) is a tool that can be used to draw lines, or segments thereof.

2-5 Regular polygons is a polygon in which all sides and angels are equal otherwise it is "irregular".

2-6 Fermat Primes: is a prime number of the form $F_m = 2^{(2^n)} + 1$ for $m = 0,1,2,\cdots$. The first five Fermat Primes are $3,5,17,257$ and 65537.

2-7 transcendental number is a number that cannot written as the root of a polynomial equation with integer coefficients.

3. Historical background:

3.1. Cubic Equation:

Cubic equation was known since the ancient times; even by the ancient Greeks and the ancient Babylonians, and also the ancient Egyptians, who dealt with the problem of doubling the cubic.

In the 11th century, the famous mathematician ***Omar Khayyam*** discovered a geometrical method to solve cubic equation which could be used to get numerical answer by intersecting a parabola with a circle, and by using this method he found cubic equation can have more than one solution.

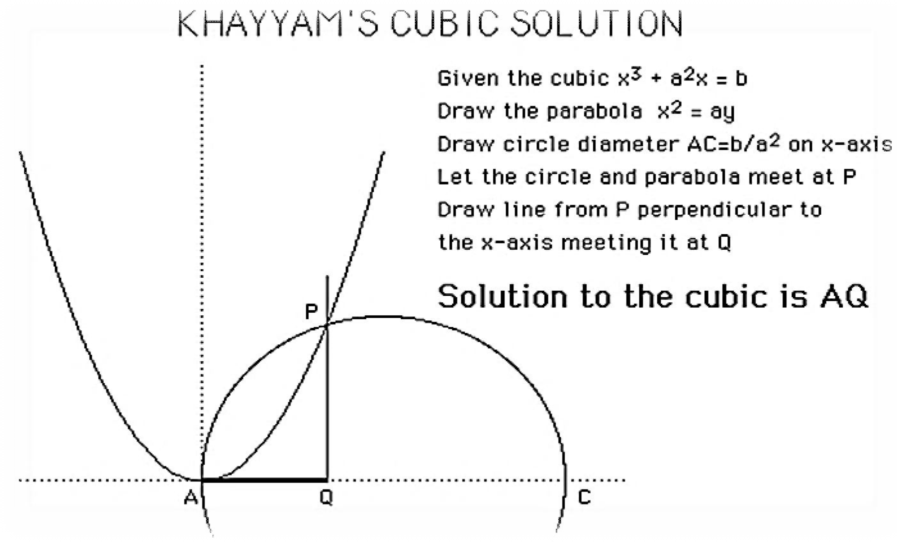

Figure1: Khayyam's method to solve cubic equation on form $x^3 + a^2x = b$, Taken from [34]

He could not find algebraic formula for the general cube but he could only solve cubic equation geometrically. He also invited other mathematician to look for algebraic formula for solving cubic equation.

Before Omar Khayyam, in the 9th century ***Al-Mahan***, after him ***Abu Jaffa Al-Kazan*** in the 10th century and after them ***Abu al-Jud ibn Laith*** in the 11th century solved cubic equation geometrically.

Also there were other Arabic and Islamic mathematicians who wrote about solving cubic equation, such as ***Al-Amir Abu Mansur Nasr bin ' Iraq*** and ***Al-Hasan ibn Al-Haythem***.

In the 12th century, the mathematician ***Sharf Al-Din Al-Tusi*** continued the extensive work of Omar al–Kayyam on a formulaic solution of the cubic equation.

He wrote an article about cubic equation entitled ***"Treatise on Equations"***, which dealt with eight types of cubic equation with nonnegative roots and five types of cubic equation which may have negative roots. The aim of this article was the study of the curves by means of equations, which represents an essential contribution to algebra and anticipated analytic geometry centuries before Rene Decartes.

Thanks in part to their work, Italian mathematicians found in the 16th century new formulas to solve the cubic and quartic equations. In the Renaissance Italy of the early 16th century ***Scipione dell Ferro*** was the first mathematician who found a general formula for solving cubic equation of the form:

$$x^3 + ax = m$$

His work was never published because he did not want to share these secrets, but when he was near death he gave these secrets to his student ***Antonio Maria Fior***. Because of that Fior started to boast that he was able to solve cubic equations by himself. So, in 1535 Fior participated on ***a contest*** with mathematician called ***Tartaglia,*** whose real name was ***Nicalo Fontana.*** Such contests were common in the Renaissance Italy.

Fior posed the following thirty problems to Tartaglia (as he was more widely known), one of these problems was:

"There was a tree of height of 12, which was broken in two parts at such a point that the height of the part which was left standing was the cube root of the length of the part that was cut a way. What was the height of the part that was left standing?"

$$[\, x^3 + x = 12 \,]$$

Ten days before the contest Tartaglia found the method to solve two types of cubic equation. The first type was " the cubes and things together are equal to some numbers: " x^3 and mx together are equal n " (in modern notation $x^3 + mx = n$), second type was " the cubes and the squares equal to numbers : x^3 and px^2 together are equal to q " (in modern notation $x^3 + px^2 = q$) .

Tartaglia could solve the equation of the form $x^3 + mx = n$ (the first type), and he could also solve the equation of the $x^3 + px^2 = q$ (the second type) by transforming it to the first type but Fior could solve only the equation of the form $x^3 + mx = n$ (the first type)**.**

Cardano took this knowledge from Tartaglia and promised that he would leave that as a secret but he published this knowledge in his Ars Magna (Latin**:"*The Great Art*** "). So, this knowledge is known today as Cardano's formula**.**

3.1.1. Tartaglia's Solution to the Cubic Equation (as in the Ars Magna):

The general cubic equation

$$y^3 + Ay^2 + By + C = 0 \qquad (1)$$

Can be reduced to the simpler form

$$\mathrm{X}^3 + \mathrm{cX} = \mathrm{d} \text{ (i.e. without the square term)} \qquad (2)$$

by using substitution $y = X - \frac{A}{3}$. So for solving any cubic equation, it suffices to solve ones of the form $X^3 + cX = d$. Cardano describes the following method for solving equation (2) in the Ars Magna (The Great Art)

1 – Find u and v so that

$$\text{(a)} \quad u - v = d$$

$$\text{(b)} \quad uv = \left(\frac{c}{3}\right)^3$$

2 - The solution to the equations (a) and (a) for u and v are given by

$$\text{(a)} \; u = \sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3} + \frac{d}{2}$$

$$\text{(b)} \; v = \sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3} - \frac{d}{2}$$

3 - The root to equation (2) is given by

$$X = \sqrt[3]{u} - \sqrt[3]{v}$$

4 – The other two roots, that are not given in the Ars Magna, are:

$$X = -\frac{1}{2}\left(\sqrt[3]{u} - \sqrt[3]{v}\right) \pm \frac{\sqrt{-3}}{2}\left(\sqrt[3]{u} + \sqrt[3]{v}\right).$$

3.1.2. Example (from Cardano's Ars Magna): find the roots of $X^3 + 6X = 20$.
Note that $c = 6$ and $d = 20$.
For solving this equation set

$u = \sqrt{\left(\frac{20}{2}\right)^2 + \left(\frac{6}{3}\right)^3} + \frac{20}{2}$ and $v = \sqrt{\left(\frac{20}{2}\right)^2 + \left(\frac{6}{3}\right)^3} - \frac{20}{2}$ hence $u = \sqrt{108} + 10$, and $v = \sqrt{108} - 10$.

The root given in the Ars Magna is then

$$X = \sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10} = 2$$

But the other two roots that are not given in the Ars Magna would be given by

$$X = -1 \pm \frac{\sqrt{-3}}{2}\left(\sqrt[3]{\sqrt{108} + 10} + \sqrt[3]{\sqrt{108} - 10}\right) = -1 \pm 3i\,.$$

Therefore, when he tried to solve problems about cubic equation he realized that he needed expressions of the form $a + b\sqrt{-1}$ in order to write down a general formula for solving cubic equation. In the 18^{th} century, ***Leonhard Euler*** introduced the symbol i to denote $\sqrt{-1}$ for complex unit and he also devised a graphical representation of it.

3.2. Quartic equation:

In 1545 Cardano published the book entitled The Ars Magna which was considered as the beginning of Algebra. Five years before him ***Lodovico Ferrari*** discovered formula to solve quartic equation , and about the same time ***Rene Descartes*** also found the similar formula for solving the quartic equation.

3.2.1. Ferrari's solution to the quartic equation (as in the Ars Magna):

The general form of quartic equation

$$y^4 + Ay^3 + By^2 + Cy + D = 0$$

can be reduced to simpler form

$$X^4 + aX^2 + bX + c = 0 \qquad (1)$$

by using the substitution $y = X - \frac{A}{4}$. So, for solving any quartic equation it is enough to consider those of the form (1) (i.e. without the cube term).

For solving equation (1), first rewrite it as:

$$X^4 + aX^2 = -bX - c$$

Add $aX^2 + a$ to both sides in order to get a perfect square

$$X^4 + 2aX^2 + a^2 = aX^2 - bX + a^2 - c.$$

$$\text{So} \quad (X^2 + a)^2 = aX^2 - bX + a^2 - c\,. \qquad (2)$$

Note that $(X^2 + a + z)^2 = X^4 + 2aX^2 + a^2 + 2X^2z + 2az + z^2$.

To balance out we will add $2X^2z + 2az + z^2$ to the RHS of (2). Our goal is to write the RHS as a perfect square too. Thus

$$(X^2 + a + z)^2 = (aX^2 - bX + a^2 - c) + 2X^2z + 2az + z^2$$

Now, grouping the corresponding terms on the RHS gives

$$(X^2 + a + z)^2 = (2z + a)X^2 - bX + (a^2 - c + 2az + z^2) \qquad (3)$$

Next, we can try to choose z so that the RHS of equation (3) is a perfect square. Then RHS of equation (3) has to be of the following form

$$\left(\left(\sqrt{2z + a}\right)X \pm \sqrt{a^2 - c + 2az + z^2}\right)^2 \qquad (*)$$

Expanding $(*)$ (i.e. squaring$(*)$) shows that the RHS should be of the form

$$(2z + a)X^2 \pm \left(2\sqrt{2z + a}\sqrt{a^2 - c + 2az + z^2}\right)X + (a^2 - c + 2az + z^2) \quad (**)$$

Equating RHS of equation (3) and (**) will give

$$-b = \pm 2\sqrt{2z + a}\sqrt{a^2 - c + 2az + z^2}$$

(4)

Squaring both sides, we get

$$b^2 = 4(2z + a)(a^2 - c + 2az + z^2)$$

Regrouping, this gives

$$4(2z + a)(a^2 - c + 2az + z^2) - b^2 = 0 \qquad (5)$$

Solving a cubic equation in z (equation (5)) will force the RHS of equation (3) to be a perfect square. Then we can use methods from the Ars Magna for solving cubic equations to find roots, and can write equation (3) as:

$$(\mathrm{X}^2 + \mathrm{a} + \mathrm{z})^2 = \left(\left(\sqrt{2\mathrm{z} + \mathrm{a}}\right)\mathrm{X} \pm \sqrt{\mathrm{a}^2 - \mathrm{c} + 2\mathrm{az} + \mathrm{z}^2}\right)^2 \qquad (6)$$

Then we take a square root of equation (6) and we get:

$$(\mathrm{X}^2 + \mathrm{a} + \mathrm{z}) = \pm\left(\left(\sqrt{2\mathrm{z} + \mathrm{a}}\right)\mathrm{X} \pm \sqrt{\mathrm{a}^2 - \mathrm{c} + 2\mathrm{az} + \mathrm{z}^2}\right) \qquad (7)$$

At the end we can use quadratic formula to get four solutions to the original quartic equation.

3.2.2. Example given in Ars Magna: Find the roots of equation $X^4 - 12X + 3 = 0$, for which $a = 0$, $b = -12$ and $c = 3$.

By using equation (5) we can get the cubic equation in z

$$0 = 4(2z + 0)(0^2 - 3 + z^2) - (-12)^2$$

$$0 = 8z(-3 + z^2) - 144$$

$$0 = 8z^3 - 24z - 144$$

When we use the Tartaglia's method, we get $z = 3$ as the root of the above cubic equation. From equation (4) we get:

$$-b = \pm 2\sqrt{2 * 3 + 0}\sqrt{0 - 3 + 0 + (3)^2} = \pm 2\sqrt{6}\sqrt{6} = \pm 12$$

Since $-b = 12$, the sign in equation (6) is should be plus. For solving the original quartic, we apply this to the equation (7)

$$(X^2 + 0 + 3) = \pm\left(\left(\sqrt{2 * 3 + 0}\right)X \pm \sqrt{0^2 - 3 + 2 * 0 * 3 + (3)^2}\right)$$

$$= \pm\left(\sqrt{6}X + \sqrt{-3+9}\right)$$

Therefore $X^2 + 3 = \pm\left(\sqrt{6}X + \sqrt{6}\right)$

By using quadratic formula we get:

$$X = \frac{\sqrt{6} \pm \sqrt{4\sqrt{6}-6}}{2} \quad \text{or} \quad X = \frac{-\sqrt{6} \pm \sqrt{4\sqrt{6}-6}}{2}$$

Ferrari did not write the last to solutions which are not real numbers, because the complex numbers were not discovered until the 17th century, and as we said before they were discovered by Leonhard Euler.

3.2.3. Ferrari's Method In modern notation:

The general form for quartic equation is

$ax^4 + bx^3 + cx^2 + dx + e = 0$, where $a \neq 0$ (1)

- If $a \neq 1$ we divide, we simplify the given equation by dividing $(b, c, d$ and $e)$ by a. Our resulting equation is:

$$X^4 + \frac{b}{a}X^3 + \frac{c}{a}x^2 + \frac{d}{a}X + \frac{e}{a} = 0 \qquad (2)$$

- Now we define some variables that are needed to solve equation (2) (quartic equation)

$$f = c - \left(\frac{3}{8}b^2\right)$$

$$g = d + \left(\frac{1}{8}b^3\right) - \left(\frac{1}{2}bc\right)$$

$$h = e - \left(\frac{3}{256}b^4\right) + \left(\frac{1}{16}b^2c\right) - \left(\frac{1}{4}bd\right)$$

- We put the numbers f, g and h into the following cubic equation:

$$X^3 + \left(\frac{1}{2}f\right)X^2 + \left(\frac{f^2-4h}{16}\right)X - \left(\frac{1}{64}g^2\right) = 0 \qquad (3)$$

- We evaluate the a, b, c and d coefficients for solving equation (3)

$$b = \frac{1}{2}f,\ c = \frac{f^2-4h}{16} \text{ and } d = \frac{-g^2}{64}$$

- Then we solve the following cubic equation by finding the roots:

$$X^3 + bX^2 + cX + d = 0 \qquad (4)$$

- For getting three roots of equation (4) we can use Tartaglia's method (or a simpler method if possible).

. Let p and q be the square roots of any two non-zero roots $(X_1, X_2$ and $X_3)$

. Let $r = \frac{-g}{8pq}$ and $s = \frac{b}{4a}$

- Now we have everything which can help us solve quartic equation and find four roots:

$$X_1 = p + q + r - s$$
$$X_2 = p - q - r - s$$
$$X_3 = -p + q - r - s$$
$$X_4 = -p - q + r - s$$

3.2.4. Example: $3X^4 + 6X^3 - 123X^2 - 126X + 1080 = 0 \ (a \neq 1)$

We simplify the above equation by dividing all terms by a. Because $3 \neq 1$, the equation becomes:

$$X^4 + 2X^3 - 41X^2 - 42X + 360 = 0$$

Where $a = 1,\ b = 2,\ c = -41,\ d = -42$ and $e = 360$

Next let us compute the variables needed to solve equation (1) :

$$f = c - \left(\frac{3}{8}b^2\right) = -41 - \left(\frac{3}{8} \times (2)^2\right) = -42.5$$

$$g = d + \left(\frac{1}{8}b^3\right) - \left(\frac{1}{2}bc\right) = d + \left(\frac{1}{8} \times (2)^3\right) - \left(\frac{1}{2} \times 2 \times -41\right)$$

$$= -42 + \left(\frac{8}{8}\right) - \left(\frac{1}{2} \times -82\right) = 0$$

$$h = e - \left(\frac{3}{256}b^4\right) + \left(\frac{1}{16}b^2c\right) - \left(\frac{1}{4}bd\right)$$

$$= e - \left(\frac{3}{256} \times (2)^4\right) + \left(\frac{1}{16} \times (2^2) \times -41\right) - \left(\frac{1}{4} \times 2 \times -42\right)$$

$$= \left(\frac{3}{256} \times 16\right) + \left(\frac{1}{16} \times 4 \times -41\right) - \left(\frac{1}{4} \times 2 \times -42\right)$$

$= 360 - 0.1875 - 10.25 + 21 = 370.56$

We substitute the numbers f, g and h into the following cubic equation:

$$X^3 + \left(\frac{1}{2}f\right)X^2 + \left(\frac{f^2 - 4h}{16}\right)X + \left(\frac{1}{64}g^2\right) = 0$$

$$X^3 + \left(\frac{1}{2} \times -42.5\right)X^2 + \left(\frac{(-42.5)^2 - 4 \times 370.5625}{16}\right)X + \left(\frac{1}{64} \times (0)^2\right) = 0$$

$X^3 - 21.25X^2 + 20.25X = 0$

we can solve the above equation by taking X as a common factor:

$$X(X^2 - 21.25X + 20.25) = 0$$

Then the first root $X_1 = 0$ and by using quadratic formula on the last quadratic equation we can find the second and the third root to the cubic equation:

$$X^3 - 21.25X^2 + 20.25X = 0$$

Then $x = \dfrac{21.25 \pm \sqrt{(-21.25)^2 - 4 \times 1 \times 20.25}}{2 \times 1} = \dfrac{21.25 \pm \sqrt{451.5625 - 81}}{2}$

$\therefore\ X_2 = \frac{21.25+19.25}{2} = 20.25$ and $X_3 = \frac{21.25-19.25}{2} = \frac{2}{2} = 1$

Let p and q be the square roots of non-zero roots $(X_1, X_2\ or\ X_3)$

$p = \sqrt{1} = 1$

$q = \sqrt{20.25} = 4.5$

$r = \frac{-g}{8pq} = \frac{0}{8\times1\times4.5} = 0$

$s = \frac{b}{4a} = \frac{6}{4\times3} = 0.5$

Then the solutions of the quartic equation are:

$X_1 = P + q + r - s = 1 + 4.5 + 0 - 0.5 = 5$

$X_2 = P - q - r - s = 1 - 4.5 - 0 - 0.5 = 3$

$X_3 = -P + q - r - s = -1 + 4.5 - 0 - 0.5 = -4$

$X_3 = -P + q + r - s = -1 - 4.5 + 0 - 0.5 = -6$

3.4. Quintic equation:

In the previous sections, we have seen how it is possible to solve cubic and quartic equations using only the four basic operations $(+,-,\times,\div)$ and taking roots of numbers (i.e. $\sqrt[n]{\ }$). Equations that can be solved using these operations are called solvable by radicals.

The important question that arises is: can we solve quintic equation by using radicals too.

Unlike the equations of degree smaller than five, the quintic equation cannot be solved by radicals.

This fact was not clear and was not known until the beginning of 19^{th}century. Norwegian mathematician ***Niles Henrik Abel*** introduced one of the most profound impossibility theorems, the Abel and Ruffini theorem also known as the Abel impossibility Theorem. It states that there is no general algebraic solution to the equations of degree $n \geq 5$ in radicals.

When he started working on quintic equation he was very young, aged only 19 (about 1821), but this work was finished in 1824. However, sadly, he died in 1829, when he was only 26 years old, of tuberculosis and malnutrition. However after his death his work was published in 1830.

Abel was not the only mathematician who worked on quintic equation. In that period also French mathematician ***Everiste Galois***, by his inimitable genius, find method to determine if the equation could be solved by radicals or no, by using sophisticated techniques which led to the founding of the group theory and Galois theory. But the proof of this theorem was published in 1846, after he died in 1832 in duel. One day before he died he felt that his death became imminent, and therefore he spent the whole night writing letters to one of his friends, which was in mathematics, and in which he outlined his ideas, and he asked him to publish them in the "Encyclopedic Journal".

Sadly, most of his work remained mysterious and incomprehensible for a long time because he did not find enough time to write in detail about all his achievements in letters to his friend at the night of his death.

In fact his life was very miserable, starting from the sudden death of his father, disregard of his works by mathematicians to his death at very young age, when he was only 21 years old.

So how we can solve the equations when $n \geq 5$ if we cannot by using radicals?

Although there are specific cases of quantic equation (or general ≥ 5) that may be solved by radicals for example: $x^5 - x^4 - x + 1 = 0$ can be solved by factorizing and written as $(x^2+1)(x+1)(x-1)^2$ or $x^5 - 3$, which has $\sqrt[5]{3}$ as a solution or $x^5 - x^3$ by taking x^3 as a common factor we get $x^3(x^2-1) = x^3(x-1)(x+1)$. However, the easiest way to solve equation when $n \geq 5$ is by using Numerical methods.

4. <u>Geometric constructions</u>:

4.1. Three famous problems with straightedge and compass constructions:

In this section we will give a short explanation about geometric constructions in order to clarify for students three famous problems, unsolvable by using straightedge and compass. These problems have captivated the minds of many mathematicians since the

ancient times. The three problems are: Squaring of a Circle, Trisecting the Angle and Doubling of a Cube.

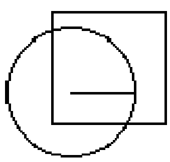

Figure 2: squaring a circle, Taken from [35]

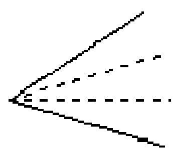

Figure 3: Trisecting the angle, Taken from [35]

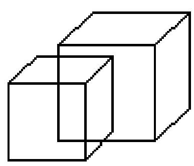

Figure 4: Doubling of a cube, Taken from [35]

These three constrictions are impossible to do using straightedge and compass, which was shown using polynomial equations. If a construction can be done, then the coordinates of the constructed point can be expressed using four basic operations $(+, -, \times, \div)$ and taking square roots.

1) ***Squaring the circle:*** This is a problem suggested by ancient mathematician. In other words, squaring of a circle means constructing a square that has the same area as a circle by using only finite number of steps with compass and straightedge. This is impossible. For this reason the expression "squaring the circle" is used to mean "doing the impossible"; in 1882 Lindemann proved that squaring of a circle was impossible in *Lindemann –Weierstrass theorem,* which proves that $\pi = 3.1415 \cdots$ is a transcendental number (not algebraic, and hence not constructible).
2) ***Angle Trisection:*** Anacient Greeks were trying to find a way to divide an angle into three equal parts. They were not able to do this using straighedge and compass. Using theory of polynomial equations it was shown that this is not possible.
3) ***Doubling the Cube:*** This problem has three different names such as "Doubling the cube", "duplicating the cube" and "Delian problem". This problem asks that if you have a cube of some side length L and volume $V = L^3$ to construct a new cube, larger than the first, with volume $2V$, i.e. the side length L$\times \sqrt[3]{2}$. The problem of doubling the cube is impossible to resolve by compass and straightedge, since $\sqrt[3]{2}$ is not a constructible number.

4.2. Constructing Regular polygons with straightedge and compass:

Another important topic related to geometric constrictions are the ***regular polygons***. The question here is if it is possible to construct all regular n-gons by using only straightedge and compass. The answer is that some regular polygons are easy to constrict using only straightedge and compass; others are impossible. In 1799 Carl Friedrich Gauss managed to find sufficient condition for constructability of regular polygons: "A regular n-gon can be constructed using only ruler and compass if n is the product of a power of 2 and any number of distinct Fermat primes" but a full proof with sufficient and necessary condition was given by Pierre Wantzel in 1837.The result is known as the Gauss-Wantzel theorem.

Gauss-Wantzel theorem states: "A regular n-gon is constructible using only ruler and compass if and only if $n = 2^k P_1 P_2 \cdots P_t$ where k and $t \in N \cup \{0\}$ and each P_i is a (distinct) Fermat Prime. By Gauss' Theorem, and the list of the first five known Fermat Prime numbers, we can list all constructible regular n-gons with $3 \leq n \leq 200$: we have that the regular n-gon is constructible if and only if n can be written as a product: $n = 2^k 3^a 5^b 17^c$ with $k \geq 0, a = 0,1, b = 0,1$ and $c = 0,1$. Consequently, n is one of the following numbers:

$3,4,5,6,8,10,12,15,16,17,20,24,30,32,34,40,48,51,60,64,68,80,85,96,102,120,128,136,160,170,192$.

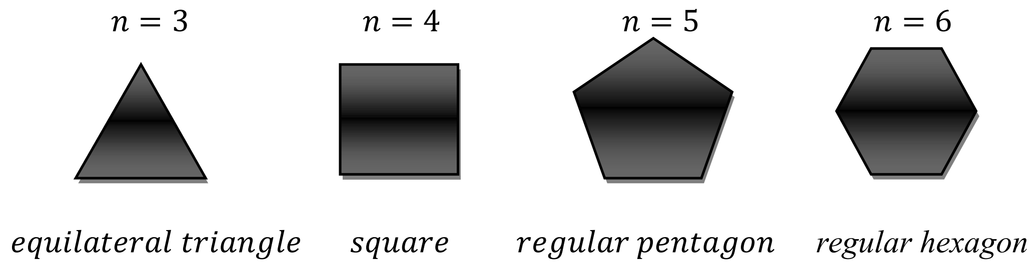

Figure 1 : Examples of Regular Polygon

5. Modern research:

Polynomial equations are still subject of modern research, especially in relation to numerical methods (see [16]). For instance, Steve Smale, a Fields medal winner, asked in his paper (see [17]) about solving equations using generally convergent purely iterative algorithms, which generalize solving equations by radicals to complex dynamics and this was answered by another Fields medalist, Curt McMullen with Peter Doyle in ([16]) showing that quintic equation can be solved in this way. Also, the Abel-Ruffini theorem, which shows that there is no solution of a general polynomial equation of degree greater than five, is still subject of current research; for instance, in [27] a new proof was given, using methods from Topology.

Students can be told that that mathematics is an active area of research and that mathematicians are still working on problems related to polynomial equations. Also, a teacher can mention Fields medal, a highest prize for mathematicians, when talking about modern research.

6. Conclusion:

In this paper we explain the solution of cubic and quartic equations (with exciting about history of their discovery). We also review modern research about this topic. All these stories can be useful for teaching mathematics in many ways.

For instance students can learn that not all problems need to have a solution. For example squaring a circle, trisection of an angle or solving quintic equation by radicals cannot be done. Students often think that in mathematics all problems are necessarily solvable and it is important for them to understand that is not always the case.

Inspiration and beauty are important aspect of mathematics. Often in teaching not enough attention is given to inspiring students, which these topics can do. History of mathematics has many inspirational stories, that can help motivate students. In the case of the history of solving cubic and quartic equations, there are also a lot of Arabic mathematicians that can serve as Role Models for students in Lybia.

Students can also learn that many topics in mathematics are interrelated. Solving polynomial equation is a good example of this, because it connects algebra, geometry constructions, complex numbers, number theory, numerical analysis, group theory. Moreover, these topics are close to what students already learn in high school.

These topics are also source of good material for gifted students. They can be introduced to a lot of advanced mathematics in an interesting and accessible way. Work with gifted students is an important issue in mathematical education, and perhaps more can be done to address this group of students in Libya.

All these information of course is helpful for teachers when teaching and for students to understand Mathematics, not only in high school but in university also.

7. Acknowledgement:

I would like to thank Professor Vladimir Božin (University of Belgrade) who read this article and gave me useful comments and suggestions.

References:

1) *Villanueva J, The Cubic and Quartic Equations: Intermediate Algebra Course, Electronic Proceedings of ICTCM, Florida Memorial College. From: http://archives.math.utk.edu/ICTCM/VOL16/C037/paper.pdf (Accessed on 1st February 2013).*
2) *Poole David, Linear Algebra: A modern Introduction, 2nd Ed, USA, Cengage Learning, 2006, pp. 650-669; p. 73.*
3) *Zill Dennis G & Dewar Jacqueline M, Precalculus with Calculus Previews, 4th Ed, Canada, Jones and Bartlett publishers, 2010, p 129.*

4) *Fraleigh John B & Beauregard Raymond A, Linear algebra, 2nd Ed, California, Addison-Wesley Publishing Company, 1990, pp. 407-410.*
5) *Bloom David M, Linear Algebra and Geometry, 1st Ed, Cambridge, Cambridge University Press, 1979, pp. 56-58.*
6) *Phillips George M, Mathematics Is Not a Spectator Sport, 1st Ed, New York, Springer, p.166.*
7) *Ross Debra Anne, Master Math: Geometry, 2nd Ed, USA, Course Technology PTR, 2009, p. 248; p. 329.*
8) *Rock Nathaniel Max, Standards-Driven Power Geometry I: Textbook and Classroom Supplement, 1st Ed, Team Rock Publishing, 2005-2006, Chapter Geometry I, Constructions, p. 2.*
9) *Dunham William, Journey through Genius: The Great Theorems of Mathematics, New York, John Wiley and Sons Inc., 1990, pp. 135-147.*
10) *Cohen David with Lee Theodore B & Sklar David, Precalculus: A problems-Oriented Approach, 6th Ed, USA, Thomson Brooks/Cole, 2005, pp. 444-445.*
11) *William L Hosch (Edited), The Britannica Guide to Algebra and Trigonometry, 1st Ed, New York, Britannic Educational Publishing, 2011, p. 182.*
12) *Ing Law Huong, The Comparison between Methods of solution for Cubic Equation in Shushu Jiuzhang and Risālah fil-barāhin 'alā masā 'il ala-Jabr Wa'l-Muqābalah, Mathematical Medley, Vol. 30, No. 2, 2003, pp. 91-101.*
13) *Hogendijk Jan P, Sharf al-Dīn al-Tūsī on The Number of Positive Roots of Cubic Equation, Historia Mathematica, Vol. 16, No 1, 1989, pp.69-85.*
14) *Kelly Brendan, Advanced Algebra with the T1-84 plus Calculator, Burlington, Brendan Kelly publishing Inc., 2007, p. 46.*
15) *Pan Victor Y, Solving a Polynomial Equation: Some History and Recent Progress, Siam Review, Vol. 39, No.2, 1997, pp. 187-220.*

16) *Doyle Peter & Mcmullen Curt, Solving The Quintic by Iteration, Acta Math, Vol. 163, No. 1, 1989, pp. 151-180.*

17) *Smale Steve, On The Efficiency Of Algorithms In Analysis, Bulletin (New Series) Of The American Mathematical Society, Vol. 13, No. 2, 1985, pp.87-121.*

18) *Boyer Carl B, A History of Mathematics, New York, John Wiley & Sons, 1968, pp. 310-557.*

19) *Smith David E, History of Mathematics, Vol. 1, New York, Dover Publications Inc, 1958, pp. 293-298*

20) *Heinrichs Heidi, Cardano-Tartaglia's Cubic Equation, Lecture Notes, 2011. From:* http://prezi.com/nu9noxlra5ax/cardano-tartaglia-cubic-equations/ *(Accessed on 16th February 2013).*

21) *Fauvel John & Gray Jeremy (Editors), The History of Mathematics: A reader, Palgrave Macmillan, 1987, PP. 254-255.*

22) *Edanur Bayam-Büşra Çam, Several Analytic Techniques of Solving Cubic Equation with Luddhar's New Method: Math 492 Graduation project I, Lecture Notes, 2011. From:* http://mcs.cankaya.edu.tr/proje/2011/yaz/eda_busra/Sunum.pdf *(Accessed on 18th February 2013).*

23) *A brief History of The Solution of The Cubic and Quartic Equation. From:* http://dwick.org/pages/cubicquartic.pdf *(Accessed on 11th March 2013)*

24) *Guilbeau Lucye, The history Of The Solution Of The Cubic Equation, Mathematical Association Of America, Vol. 5, No. 4,1930, pp. 8-12.*

25) *Johnston William & McAllister Alex M, A Transition to Advanced Mathematics: A Survey Course, Oxford, Oxford University press Inc., 2009, pp.218-224; p.526.*

26) *Solving Quartic Equations. From:* http://jwilson.coe.uga.edu/EMAT6680Fa09/Davenport/Solving%20Quartic%20Equations.pdf *(Accessed on 12th May 2013)*

27) *Żoladek Henryk, The Topological Proof of Abel-Ruffini theorem, Journal of The Juliusz Schauder Center: Topological Methods in Nonlinear Analysis, Vol.16, 2000, pp. 253-265.*

28) *Rothman Tony, Genius and Biographers: The Fictionalization of Evariste Galois, Mathematical Association of America, Vol. 89, 1982, pp. 84-106.*

29) *Rosen Michael I, Niels Hendrik Abel and Equation of Fifth Degree, Mathematical Association of America, Vol. 102, No. 6, 1995, pp. 495-505.*

30) *Süli Endre & Mayers David, An introduction to Numerical Analysis, 1st Ed, Cambridge, Published by The Syndicate of The University of Cambridge, 2003, pp. 1-2.*

31) *Jacobs Conrad, Invitation to Mathematics, United Kingdom, Princeton University Press, 1992, pp. 11-16.*

32) *Irwin Sarah, Impossible construction, Lecture Note, 2012. From:* http://prezi.com/ja8lyjlopf2g/impossible-constructions/ *(Accessed on 18th June 2013).*

33) *Laczkovich Miklós, Conjecture and Proof, Cambridge, Cambridge University Press, 2001, p. 7.*

34) *Difference Equation and Recursive Relations. From:* http://fibonacci.math.uri.edu/~kulenm/diffeqaturi/m381f00fp/karen/karenmp.htm *(Accessed on 13th July 2013).*

35) *"Impossible" Geometric Constrictions. From:* http://mathforum.org/dr.math/faq/faq.impossible.construct.html *(Accessed on 1st August 2013).*