On The Global Well-Posedness Result of A Class Of Boussinesq-Navier-Stokes Systems

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Abstract:

This paper deals with the global well-posedness of the bidimensionnel Boussinesa system which couples the incompressible Euler equation with fractional diffusion for the velocity and a transport diffusion for the temperature with initial data

$$v^0 \in H^1(\mathbb{R}^2) \cap \dot{W}^{1,p}(\mathbb{R}^2), 2 < p < \infty, \theta^0 \in L^2(\mathbb{R}^2) \cap B^0_{\infty,1}(\mathbb{R}^2).$$

Keywords. Global well− posedness, Besov spaces, calcul differential.

1. Introduction:

Generalized Navier-Stokes-Boussinesq equations (GNSB) for the incompressible fluid flows in $\mathbb{R}^2$ have the form

$$\begin{cases} \partial_t v + v.\nabla v + \nabla p = F(\theta) + Kv \\ \partial_t \theta + v.\nabla\theta = H(\theta), \qquad div\, v = 0 \\ v(t=0) = v^0, \qquad \theta(t=0) = \theta^0 \end{cases}$$

where $v = (v_1, v_2);\ v_j = v_j(x,t), j = 1,2, (x,t) \in \mathbb{R}^2 \times [0,\infty)$ the velocity field, $p = p(x,t)$ is the scalar pressure and $\theta(x,t)$ is the scalar temperature. F and H are given smooth functions. The following cases for the linear dissipative operator K will be discussed : $Kv = 0$ (non dissipation), $Kv = v$ (linear function) and $Kv = \Delta v$ (viscosity).

The Boussinesq system has important roles in the atmospheric sciences for $F(\theta) = \theta e_2, e_2 = (0,1)$ (see for example [4]).

Before discussing the mathematical aspect of our model with general F we will first focus on the special case $F(\theta) = \theta e_2, e_2 = (0,1)$. The simplest model for the mathematical study is the fully viscous model i.e when $Kv = \Delta v, H(\theta) = \Delta\theta$. The global well-posedness result can be obtained in this case. On the other hand, the regularity question of the case when $F(\theta) = (0,\theta), Kv = 0$ and $H(\theta) = 0$ is an outstanding open problem in the mathematical fluid mechanics (See [10, 16, 19] for studies in this direction).

We note that in space in dimension two, the vorticity is defined by the scalar $w = \partial_1 v^2 - \partial_2 v^1$ solves the equation

$$\partial_t w + v.\nabla w = \partial_1\theta.$$

The main difficulty is that to get an L^∞ estimate on w which is crucial to prove global.

existence of smooth solutions , one need to estimate $\int_0^t \|\partial_1\theta(\tau)\|_{L^\infty}\, d\tau$.

For the fully viscous i.e $Kv = \Delta v$ and $H(\theta) = 0$: This system can be seen as an hyperbolic quasi-linear system and thus it is locally well-posed in Sobolev spaces H^s with $s > 2$. The global well-posedness results were recently established in various functional spaces. Chae [9] proved the global existence and uniqueness for initial data $(v^0, \theta^0) \in H^s$, with $s > 2$. This result was improved by [20] for less regular initial data i.e $v^0, \theta^0 \in H^s$ with $s > 0$. In [13], Danchin and Paicu proved the uniqueness result in the energy space

L^2. According to a recent work of [14] one can construct global unique solution when the dissipation acts only in the horizontal $Kv = \partial_{11} v$. instead of Δv.

More Recently Hmidi, Keraani and Rousset [23] proved for $H(\theta) = 0$ the global well−posedness for fractional diffusion $Kv = -|D|v$ with initial data $v^0 \in H^1 \cap \dot{W}^{1,p}$ with $2 < p < \infty$ and $\theta^0 \in L^2 \cap B^0_{\infty,1}$ where the dissipation $|D|$ defined by

$$\mathcal{F}(|D|g)(\xi) = |\xi|(\mathcal{F}g)(\xi).$$

They used the smoothing effects for the quantity $(w - \mathcal{R}\theta)$ where $\mathcal{R}$ is a Riesz transform which yield the crucial estimate on the Lipschitz norm of the velocity and by using it to propagate the L^pnorm of the vorticity.
In this paper we address the question of global existence in the case where the dissipation occurs in the velocity equation and $H(\theta) = 0$ i.e we focus on the system

$$(1.1) \qquad \begin{cases} \partial_t v + v.\nabla v + |D|v + \nabla p = F(\theta) \\ \partial_t \theta + v.\nabla \theta = 0, \qquad div\, v = 0 \\ v(t=0) = v^0, \qquad \theta(t=0) = \theta^0. \end{cases}$$

Here F is a vector valued function depend on $\theta(x,t)$ satisfies $F(0) = 0$ and $F(\theta) \in C^2(\mathbb{R}, \mathbb{R}^2)$. If we take $\theta = 0$, then the system (1.1) is reduced to the well-known $2D$ incompressible Euler Boussinesq system.

It is well known that this system is globally well-posed in $H^s, s > 2$. The main argument for globalization is the Beale-Kato-Majda criterion [5] ensuring that the development of finite -time singularities is related to the blow-up of the L^∞norm of the vorticity. Vishik in [25] has extended the global existence of strong solutions result to initial data lying in the spaces $B_{p,1}^{1+\frac{2}{p}}$. Notice that these spaces have the same scaling as Lipschitz functions and the BKM criterion cannot be used.

Let us now discuss briefly the difficulties for $Kv = -|D|^\alpha v, \alpha < 2$ and $H(\theta) = 0$. We write the system under the vorticity temperature formulation as follows

$$\begin{cases} \partial_t w + v.\nabla w + |D|^\alpha w = \partial_1\big(F_2(\theta)\big) - \partial_2\big(F_1(\theta)\big) \\ \partial_t\theta + v.\nabla\theta = 0, \qquad div\, v = 0 \\ w(t=0) = curl\, v^0, \qquad \theta(t=0) = \theta^0. \end{cases}$$

Taking the L^2 scalar product, we get,

$$\|w(t)\|_{L^2}^2 + \int_0^t \|w(\tau)\|_{\dot{H}^{\frac{\alpha}{2}}}^2 d\tau \le \|w^0\|_{L^2}^2 + \|\nabla F\|_{L^\infty} \int_0^t \|\theta(\tau)\|_{\dot{H}^{1-\frac{\alpha}{2}}}^2 d\tau$$

and ,

$$\|\theta(t)\|_{L^2} = \|\theta^0\|_{L^2}.$$

For $\alpha = 2$, we can obtain a bound for w in $L^\infty_{loc}(\mathbb{R}_+, L^2) \cap L^2_{loc}\big(\mathbb{R}_+, \dot{H}^1\big)$. For $1 < \alpha < 2$, there is no obvious a priori estimate on $\|\theta\|_{\dot{H}^{1-\frac{\alpha}{2}}}$ to estimate the vorticity and we will use the idea of [23] consisting in the use of the smoothing effect on the distance between the vorticity w and the Riesz transform on the temperature $\mathcal{R}\theta$. We intend then to use this approach to treat the general case of $F \neq 0$ and $\alpha = 1$, we consider (1.1) for which the vorticity form from of the system is:

$$\begin{cases} \partial_t w + v.\nabla w + |D|w = \partial_1\big(F_2(\theta)\big) - \partial_2\big(F_1(\theta)\big) \\ \partial_t \theta + v.\nabla\theta = 0 \\ div\, v = 0 \\ w(t=0) = curl\, v^0 \quad ,\theta(t=0) = \theta^0. \end{cases}$$

This is the crucial case in the sense that the again of one derivative by the diffusion term roughly compensates exactly the loss of one derivative in θ in the vorticity equation and has the same order as the convection term.

The main result of this paper is to extend the results of [23] for general term F and is a global well-posedness result of the system (1.1). It reads as follows (see section 2 for the definitions and the basic properties of Besov spaces).

Theorem 1.1.

Let v^0 be a divergence-free vector field such that $v^0 \in H^1 \cap \dot{W}^{1,p}$ with $2 < p < \infty$ and $\theta^0 \in L^2 \cap B^0_{\infty,1}$ .Assume that $F \in C^2(\mathbb{R}, \mathbb{R}^2)$. Then the system (1.1) has a unique global solution (v, θ) such that

$$v \in L^\infty_{loc}\big(\mathbb{R}_+, H^1 \cap \dot{W}^{1,p}\big) \cap L^1_{loc}\big(\mathbb{R}_+; B^1_{\infty,1}\big), \qquad \theta \in L^\infty_{loc}\big(\mathbb{R}_+; L^2 \cap B^0_{\infty,1}\big).$$

Let us say a few words about the main difficulties in the proof of Theorem 1.1. First we construct a function Γ as follows : applying $F_i(\theta)$ with $i = 1, 2$ to the temperature equation in the second equation of (1.1), we get

$$(\partial_t + v.\nabla)F_i(\theta) = 0$$

and acting the Riez transform $\mathcal{R}_i = \frac{\partial_i}{|D|}$, $i = 1,2$ on the previous equation,

$$\partial_t \mathcal{R}_1 F_2(\theta) + v.\nabla \mathcal{R}_1 F_2(\theta) = -[\mathcal{R}_1, v.\nabla] F_2(\theta)$$

and

$$\partial_t R_2 F_1(\theta) + v.\nabla \mathcal{R}_2 F_1(\theta) = -[\mathcal{R}_2, v.\nabla] F_1(\theta).$$

Now, if we denote $\Gamma \coloneqq w - \mathcal{R}_1 F_2(\theta) + \mathcal{R}_2 F_1(\theta)$, then we directly have

$$\partial_t \Gamma + v.\nabla\Gamma + |D|\Gamma = [\mathcal{R}_1, v.\nabla] F_2(\theta) - [\mathcal{R}_2, v.\nabla] F_1(\theta). \tag{1.2}$$

The main difficulty then to prove our Theorem is to evaluate in a sufficiently sharp way the commutator $[\mathcal{R}_i, v.\nabla]$ between the Riesz transform and the convection operator, where the commutator $[\mathcal{R}_i, v.\nabla] F(\theta)$ is defined by

$$[\mathcal{R}_i, v.\nabla] F(\theta) = \mathcal{R}_i(v.\nabla F(\theta)) - v.\nabla \mathcal{R}_i F(\theta)$$

see section 4 of the paper.

As we will see in our proof, we can obtain a bound on the quantity Γ . More precisely we have $\Gamma \in L^{\rho}_{loc}(\mathbb{R}_+, B^0_{\infty,1})$, $\forall \rho < \frac{p}{2}$, we refer to Proposition 5.6.

The rest of this paper is organized as follows. In section 2 we present some notations and recall some important preliminary results as preparation. In section 3, we give some estimates for a transport-diffusion models. In section 4, we collect some properties of Riesz operator $\mathcal{R}_i = \frac{\partial_i}{|D|}$, $\forall i = 1,2$ and finally we discuss in section 5 the proof of our main result.

2. Preliminaries

In this section, we introduce some notations and definitions of Besov space and also recall some well-known results about the Littlewood-Paley decomposition used later. Let us begin with notations.

- For any positive G and H, the notation $G \lesssim H$ means that there exists a positive constant C such that $G \lesssim CH$.

- For any tempered distribution g, $\mathcal{F}(g)$ denote the Fourier transform of g with

$$\mathcal{F}(g) = \int_{\mathbb{R}^2} g(x)e^{-ix\xi}\, d\xi.$$

- For any pair of operator C and D on some Banach space $\mathcal{A}$, the commutator $[C,D]$ is defined by $CD - DC$.
- We denote by $\dot{W}^{1,p}$ with $1 \le p \le \infty$ the space of distribution f such that $\nabla f \in L^p$.

We introduce now Littlewood-Paley decomposition and the definition of Besov spaces. Given two nonnegative radial functions $\chi \in \mathcal{D}(\mathbb{R}^2)$ and $\varphi \in \mathcal{D}(\mathbb{R}^2 / \{0\})$ such that

$$\chi(\xi) + \sum_{j\ge 0} \varphi\left(2^{-j}\xi\right) = 1, \qquad \forall \xi \in \mathbb{R}^2,$$

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \qquad \forall \xi \in \mathbb{R}^2 / \{0\},$$

$$|p-j| \ge 2 \Rightarrow supp\ \varphi(2^{-p}.) \cap upp\ \varphi\left(2^{-j}.\right) = \phi,$$

$$j \ge 1 \Rightarrow supp\ \chi \cap upp\ \varphi\left(2^{-j}.\right) = \phi.$$

Set $\varphi_j(\xi) = \varphi\left(2^{-j}\xi\right)$ and let $h = \mathcal{F}^{-1}\varphi$ and $\bar{h} = \mathcal{F}^{-1}\chi$. Define the frequency operators Δ_j and $S_j$by

$$\Delta_j f = \varphi\left(2^{-j}D\right)f = 2^{2j} \int_{\mathbb{R}^2} h\left(2^j y\right) f(x-y)dy, \qquad j \ge 0,$$

$$S_j f = \chi\left(2^{-j}D\right)f = \sum_{-1\le p\le j-1} \Delta_p f = 2^{2j} \int_{\mathbb{R}^2} \bar{h}\left(2^j y\right) f(x-y)dy,$$

$$\Delta_{-1} f = S_0 f, \qquad \Delta_j f = 0,\ \ j \le -2.$$

Recall now the following definition of general Besov spaces.

2.1 Definition:

Let $s \in \mathbb{R}$ and $1 \leq p \leq \infty$. The inhomogeneous Besov space $B^s_{p,r}$ defined by

$$B^s_{p,r} = \left\{ f \in S \quad (\mathbb{R}^2) : \|f\|_{B^s_{p,r}} < \infty \right\},$$

where

$$\|f\|_{B^s_{p,r}} := \begin{cases} (2^{js}\left\|\Delta_j f\right\|_{L^p})_{l^r} + \|S_0 f\|_{L^p}, & r < \infty \\ sup_{j\geq 0} 2^{js}\left\|\Delta_j f\right\|_{L^p} + \|S_0 f\|_{L^p}, r = \infty. \end{cases}$$

The homogeneous norm

$$\|f\|_{\dot{B}^s_{p,r}} := \begin{cases} (2^{js}\left\|\dot{\Delta}_j f\right\|_{L^p})_{l^r}, & r < \infty \\ sup_{j\in\mathbb{Z}} 2^{js}\left\|\dot{\Delta}_j f\right\|_{L^p}, & r = \infty\,. \end{cases}$$

If $s > 0$, then $B^s_{p,r} = L^p + \dot{B}^s_{p,r}$ and $\|f\|_{B^s_{p,r}} \leq \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,r}}$. We refer to [3, 18] for detail. If $p = r = 2$, $\dot{B}^s_{2,2}$ is equivalent to the homogeneous Sobolev spaces $\dot{H}^s$ which is defined below.

2.2 Definition:

Let $s \in \mathbb{R}$; the Sobolev homogeneous spaces $\dot{H}^s$ is the space of a tempered distribution f such that $f \in L^1_{loc}$ and satisfies

$$\|f\|^2_{\dot{H}^s} := \int_{\mathbb{R}^2} |\xi|^{2s} |\mathcal{F}(f(\xi))|^2 d\xi < \infty.$$

The following definition gives the mixed time-space Besov space dependent on Littlewood-Paley decomposition (see [7]).

2.3 Definition :

Let $T > 0$ and $\rho \geq 1$, we denote by $L_T^\rho B_{p,r}^s$ the space of distributions f such that

$$\|f\|_{L_T^\rho B_{p,r}^s} := \left\| \left(2^{qs} \|\Delta_q f\|_{L^p} \right)_{l^r} \right\|_{L_T^\rho} < +\infty.$$

Besides the usuel mixed space $L_T^\rho B_{p,r}^s$, we also need Chemin-Lerner spaces $\tilde{L}_T^\rho B_{p,r}^s$ which is defined as the set of all distributions f satisfying

$$\|f\|_{\tilde{L}_T^\rho B_{p,r}^s} := \left\| 2^{qs} \|\Delta_q f\|_{L_T^\rho L^p} \right\|_{l^r} < +\infty.$$

The relation between these spaces are detailed as below, which is a direct consequence of the Minkowski inequality. Let $s \in \mathbb{R}, \mathcal{E} > 0$ and $(p, r, \rho) \in [1, +\infty]^3$. Then we have the following embeddings

$$L_T^\rho B_{p,r}^s \hookrightarrow \tilde{L}_T^\rho B_{p,r}^s \hookrightarrow L_T^\rho B_{p,r}^{s-\varepsilon}, \qquad if \qquad r \geq \rho,$$
$$L_T^\rho B_{p,r}^{s+\varepsilon} \hookrightarrow \tilde{L}_T^\rho B_{p,r}^s \hookrightarrow L_T^\rho B_{p,r}^s, \qquad if \qquad \rho \geq r.$$

For convenience, we also recall the definition of Bonys para-product formula, which gives

the decomposition of the product of two functions $f(x)$ and $g(x)$ (see. [2, 12]).

2.4 Definition:

The para-product of two functions f and g is defined as

$$T_f g = \sum_j S_{j-1} f \Delta_j g.$$

The remainder of the para-product is defined as

$$R(f, g) = \sum_{|j-\bar{j}| \leq 1} \Delta_j f \Delta_{\bar{j}} g.$$

Then Bonys para-product formula reads

$$f.g = T_f g + T_g f + R(f,g).$$

Next, let us introduce some useful lemmas which will be repeatedly used throughout this paper. We start with Bernstein inequality which is fundamental in the analysis involving Besov space (see [6]).

2.5. Lemma (Bernstein Lemma).

There exists a constant $C > 0$ such that for every $q \in \mathbb{Z}, k \in \mathbb{N}$ and for every tempered distribution u we have

$$sup_{|\alpha|=k} \left\| \partial^\alpha S_j u \right\|_{L^b} \leq C^k 2^{j\left(k+2\left(\frac{1}{a}-\frac{1}{b}\right)\right)} \left\| S_j u \right\|_{L^a} \qquad , b \geq a \geq 1$$

$$C^{-k} 2^{jk} \left\| \dot{\Delta}_j u \right\|_{L^a} \leq sup_{|\alpha|=k} \left\| \partial^\alpha \dot{\Delta}_j u \right\|_{L^a} \leq C^k 2^{jk} \left\| \dot{\Delta}_j u \right\|_{L^a}.$$

The following lemma will be needed (see [23] for a proof).

2.6. Lemma (Commutators estimates).

Let v be a smooth divergence-free vector field
and h be a smooth function then

(1) for every $j \geq -1$ and $p \in [1, \infty]$,

$$\left\| [\Delta_j, v.\nabla] h \right\|_{L^p} \lesssim \|\nabla v\|_{L^p} \|h\|_{B^{\frac{2}{p}}_{p,1}} .$$

(2) For every $s \in [-1,0]$,

$$\|v.\nabla h\|_{B^s_{2,\infty}} \lesssim \|v\|_{L^2} \|h\|_{B^{1+s}_{\infty,1}}.$$

3. Transport-diffusion models:

The goal of this paragraph to gives some estimates for a transport-diffusion models. We can find the proof of the following estimate in [1].

3.1. Proposition :

Let v be a smooth divergence-free vector field and if θ be a smooth solution of the equation,

$$\begin{cases} \partial_t \theta + v.\nabla\theta = h, \\ \theta(t=0) = \theta^0. \end{cases}$$

Then for every $p \in [1,\infty]$, we have

$$\|\theta(t)\|_{B^{-1}_{p,\infty}} \le e^{C\|v\|_{L^1_t B^1_{\infty,1}}} \left(\|\theta^0\|_{B^{-1}_{p,\infty}} + \int_0^t \|h(\tau)\|_{B^{-1}_{p,\infty}} d\tau \right).$$

3.2. Proposition :

Let v be a smooth divergence- free vector field, $\alpha \in \mathbb{R}_+$ and $(p,r) \in [1,\infty]^2$. Then there exists $C > 0$ such that for every scalar solution ζ of the equation

$$\begin{cases} \partial_t \zeta + v.\nabla\zeta + \alpha|D|\zeta = h, \\ \zeta(t=0) = \zeta^0 \end{cases}$$

satisfies

$$\|\zeta\|_{\tilde{L}^\infty_t B^0_{p,r}} \le C(\|\zeta^0\|_{B^0_{p,r}} + \|h\|_{\tilde{L}^1_t B^0_{p,r}})(1 + \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau)$$

and

$$\|\zeta(t)\|_{L^p} \le \|\zeta^0\|_{L^p} + \int_0^t \|h(\tau)\|_{L^p} d\tau).$$

We mention that the result is first proved in [25] for $\alpha = 0$ by using the special structure of the transport equation. Hmidi and Keraani [22] generalized Vishik's result for a transport diffusion equation where the dissipation term takes the form $-\alpha\Delta\zeta$. The method described in [22] can be easily adapted to our model for more details can be found in [24]. The L^p estimate are proved in [8].

The following estimates on the velocity equation is useful in the proof of the uniqueness part (see [23] for a proof).

3.3. Proposition :

Let $s \in]-1,1[$, $\rho \in [1,\infty]$ and v be a smooth divergence- free vector field. Let u be a smooth solution of the system

$$\begin{cases} \partial_t u + v.\nabla u + |D|u + \nabla p = h, \\ div\, u = 0, \\ u(t=0) = u^0. \end{cases}$$

Then we have for every $t \in \mathbb{R}_+$,

$$\|u\|_{L_t^\infty B_{2,\infty}^s} \leq Ce^{CV(t)}\left(\|u^0\|_{B_{2,\infty}^s} + \left(1 + t^{1-\frac{1}{\rho}}\right)\|h\|_{\tilde{L}_t^\rho B_{2,\infty}^{s-1+\frac{1}{\rho}}}\right),$$

where :

$$V(t) := \int_0^t \|\nabla v(\tau)\|_{L^\infty}\, d\tau.$$

4.Riesz transform and commutators:

In this paragraph, we collect some useful properties of the Riesz operator $\mathcal{R}_i = \frac{\partial_i}{|D|}$ with $i = 1,2$.

4.1. Proposition :

Let $\mathcal{R}_i = \frac{\partial_i}{|D|}$ be the Riez operator $with\ i = 1,2$. Then the following hold true.

(1) For every $p \in]1, \infty[$,

$$\|\mathcal{R}_i\|_{\mathcal{L}(L^p)} \leq 1.$$

(2) Let C be a fixed ring. Then there exists $\eta \in S$ whose spectrum does not meet the origin such that

$$\mathcal{R}_i g = 2^{qd}\eta(2^q.) * g,$$

for every g with Fourier transform supported in 2^qC In particular, $\mathcal{R}_i\Delta_q$ is uniformly bounded (with respect to $q \in \mathbb{N}$ in L^p for every $p \in [1, \infty]$.

The property (1) is a classical Calderon-Zygmund theorem (see [26] for instance) and (2) is obvious.

The following lemma is useful in dealing with the commutator terms (see [24] for a proof).

4.2. Lemma :

Let $p \in [1, \infty], f, g$ and h be three functions such that $\nabla f \in L^p,\ g \in L^\infty$ and $x\ h \in L^1$. Then

$$\|h * (fg) - f(h * g)\|_{L^p} \leq \|xh\|_{L^1}\|\nabla f\|_{L^p}\|g\|_{L^\infty}.$$

The next proposition consider the crucial commutators involving the Riesz transform $\mathcal{R}_i$.

4.3. Proposition

Let v be a smooth divergence-free vector field. Then for every smooth

scalar function θ the following estimates hold true $\forall i = 1,2$.
(1) For every $r \in]0,1[$

$$\|[\mathcal{R}_i, v]\theta\|_{H^r} \lesssim_r \|\nabla v\|_{L^2}\|\theta\|_{B^{r-1}_{\infty,2}} + \|v\|_{L^2}\|\theta\|_{L^2}.$$

(2) For every $p \in [2, \infty]$

$$\|[\mathcal{R}_i, v.\nabla]\theta\|_{B^0_{p,\infty}} \lesssim_p \|\nabla v\|_{L^p}\|\theta\|_{B^0_{\infty,\infty}} + \|v\|_{L^2}\|\theta\|_{L^2}.$$

(3) For every $p \in [2, \infty[$

$$\|[\mathcal{R}_i, v.\nabla]\theta\|_{B^0_{p,1}} \lesssim_p \|\nabla v\|_{L^p}\left(\|\theta\|_{B^0_{\infty,1}} + \|\theta\|_{L^p}\right).$$

The estimates (1) and (2) are proved in [23], while the proof of (3) can be found in [24]. Notice that this proposition is valid for any vector-valued function $F(\theta) = \left(F_1(\theta), F_1(\theta)\right)$. This fact is useful in the proof of our main result in the next section (see Propositions 5.2, 5.3 and 5.5)

5. Proof of our main result:

Throughout this section we use the notation Φ_l to denote any function of the form

$$\Phi_l = C_0 \underbrace{\exp(\dots \exp(C_0 t) \dots)}_{l-times},$$

where C_0 depend on the norms of the initial data and its value may vary from line to line up to some absolute constants. We will make an intensive use of the following trivial facts

$$\int_0^t \Phi_l(\tau)d\tau \leq \Phi_l(t) \quad \text{and} \quad \exp(\int_0^t \Phi_l(\tau)d\tau) \leq \Phi_{l+1}(t).$$

We will also use the following estimate for $F(\theta)$. Applying Taylor formula at order 1; with F vanishing at 0; we get

$$F(\theta) = \theta \int_0^1 F^{`}(\tau\theta)d\tau.$$

If $\theta^0 \in L^\infty$ and $F \in C^1(\mathbb{R}, \mathbb{R}^2)$, then we have for all $p \in [1, \infty]$,

$$\|F(\theta)\|_{L^p} \leq \|\theta\|_{L^p} \int_0^1 \left\|F^{`}(\tau\theta)\right\|_{L^\infty} d\tau.$$

Now,

$$\left\|F^{`}(\tau\theta)\right\|_{L^\infty} \lesssim sup_{|y|\leq\|\theta^0\|_{L^\infty}} \left|F^{`}(y)\right| \leq C.$$

Therefore

$$(5.1) \qquad \|F(\theta(t))\|_{L^p} \lesssim \|\theta(t)\|_{L^p}, \qquad \forall t \in \mathbb{R}_+.$$

The aim of this section is the proof of Theorem 1.1. For the sake of a clear presentation, we divide it into three subsection. An a priori bound is proven in the first subsection. The second and the third subsections proves respectively the uniqueness and the existence results, with the aid of the a priori estimates.

5.1. A priori estimates.

We shall first discuss some results about weak solutions. We will prove the following energy estimates.

Proposition 5.1.

Let $(v^0, \theta^0) \in L^2 \times L^2 \cap L^\infty$ and $F \in C^1(\mathbb{R}, \mathbb{R}^2)$. Then there exists a global weak solution of the system (1.1) in $L^\infty_{loc}(\mathbb{R}_+, L^2) \cap L^2_{loc}(\mathbb{R}_+, \dot{H}^{\frac{1}{2}}) \times L^\infty_{loc}(\mathbb{R}_+, L^2)$ such that $\forall t \in \mathbb{R}_+$,

$$\|v(t)\|^2_{L^2} + \int_0^t \|v(\tau)\|^2_{\dot{H}^{\frac{1}{2}}} d\tau \leq C_0(1 + t^2),$$

$$\|\theta(t)\|_{L^2} \leq \|\theta^0\|_{L^2}.$$

Besides if $\theta^0 \in L^p$ for some $1 \leq p \leq \infty$, we further have

$$\|\theta(t)\|_{L^p} \leq \|\theta^0\|_{L^p}.$$

Proof. The L^p -estimate for θ is a direct consequence of Proposition 3.2. For the L^2 estimate for v, by taking the L^2 inner product with v in the velocity equation, we have then

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2 + \|v(t)\|_{\dot{H}^{\frac{1}{2}}}^2 = \int_{\mathbb{R}^2} F\big(\theta(t,x)\big)v(t,x)dx$$
$$\leq \|F(\theta)\|_{L^2}\|v(t)\|_{L^2}$$
$$\leq \|\theta^0\|_{L^2}\|v(t)\|_{L^2}.$$

we have used Holder inequality, Proposition 3.2 and (5.1). On the other hand,

$$\|v(t)\|_{L^2} \leq \|v^0\|_{L^2} + \int_0^t \|F(\theta(\tau))\|_{L^2}d\tau$$
$$\leq \|v^0\|_{L^2} + \|\theta^0\|_{L^2}t.$$

Putting this inequality in the previous one yields

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2 + \|v(t)\|_{\dot{H}^{\frac{1}{2}}}^2 \lesssim \|\theta^0\|_{L^2}(\|v^0\|_{L^2} + \|\theta^0\|_{L^2}t)$$

Integrating in time, we obtain

$$\|v(t)\|_{L^2}^2 + \int_0^t \|v(\tau)\|_{\dot{H}^{\frac{1}{2}}}^2 d\tau \leq C_0(1+t^2).$$

We aim now at prove some smoothing effects on the quantity Γ through studing the equation (1.2), we start with the construction of a global weak solution.

5.2. Proposition :

Let v be a smooth divergence-free vector field of $\mathbb{R}^2$ with vorticity $w := curl\, v$ such that $v^0 \in H^1$ and $\theta^0 \in L^2 \cap L^\gamma$ with $4 < \gamma \leq \infty$ and

$F \in C^1(\mathbb{R}, \mathbb{R}^2)$. Then there exists a global weak solution for the system (1.1) such that

$$\|w(t)\|_{L^2}^2 + \int_0^t \|\Gamma(\tau)\|_{\dot{H}^{\frac{1}{2}}}^2 d\tau \leq \Phi_1(t),$$

with $\Gamma := w - \mathcal{R}_1 F_2(\theta) + \mathcal{R}_2 F_1(\theta)$.

Proof. Taking the L^2 estimate of the equation (1.2),

$$\frac{1}{2}\frac{d}{dt}\|\Gamma(t)\|_{L^2}^2 + \|\Gamma(t)\|_{\dot{H}^{\frac{1}{2}}}^2$$

$$= \int_{\mathbb{R}^2} [\mathcal{R}_1, v.\nabla]F_2\big(\theta(t,x)\big)\Gamma(t,x)dx$$

$$- \int_{\mathbb{R}^2} [\mathcal{R}_2, v.\nabla]F_1\big(\theta(t,x)\big)\Gamma(t,x)dx.$$

Using $[\mathcal{R}_1, v.\nabla]F_2(\theta) = div([\mathcal{R}_1, v]F_2(\theta))$ and $[\mathcal{R}_2, v.\nabla]F_1(\theta) = div\big([\mathcal{R}_2, v]F_1(\theta)\big)$, we find

(5.2) $$\frac{1}{2}\frac{d}{dt}\|\Gamma(t)\|_{L^2}^2 + \|\Gamma(t)\|_{\dot{H}^{\frac{1}{2}}}^2 \leq (\|[\mathcal{R}_1, v]F_2(\theta)\|_{\dot{H}^{\frac{1}{2}}} + \|[\mathcal{R}_2, v]F_1(\theta)\|_{\dot{H}^{\frac{1}{2}}})\|\Gamma(t)\|_{\dot{H}^{\frac{1}{2}}}$$

Thanks to the part (1) of Proposition 4.3 with $F(\theta)$ and Proposition 5.1, we get

$$\|[\mathcal{R}_1, v]F_2(\theta)\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|\nabla v\|_{L^2}\|F_2(\theta)\|_{B_{\infty,2}^{-\frac{1}{2}}} + \|v(t)\|_{L^2}\|F_2(\theta)\|_{L^2}$$

$$\lesssim \|w(t)\|_{L^2}\|F_2(\theta)\|_{L^\gamma} + \|v(t)\|_{L^2}\|F_2(\theta)\|_{L^2}$$

$$\lesssim \|w(t)\|_{L^2}\|\theta^0\|_{L^\gamma} + C_0(1+t)\|\theta^0\|_{L^2}$$

(5,3) $$\lesssim \|w(t)\|_{L^2}\|\theta^0\|_{L^\gamma} + C_0(1+t),$$

we have also used the Calderon-Zygmand and the embedding $L^\gamma \hookrightarrow B_{\infty,2}^{-\frac{1}{2}}$ for $\gamma > 4$. Similarly, we find

$$\|[\mathcal{R}_2, v]F_1(\theta)\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|w(t)\|_{L^2}\|\theta^0\|_{L^\gamma} + C_0(1+t). \quad (5.4)$$

Putting together (5.3) and (5.4) into (5.2), we get

$$\frac{1}{2}\frac{d}{dt}\|\Gamma(t)\|_{L^2}^2 + \|\Gamma(t)\|_{\dot{H}^{\frac{1}{2}}}^2 \lesssim \big(\|w(t)\|_{L^2}\|\theta^0\|_{L^\gamma} + C_0(1+t)\big)\|\Gamma(t)\|_{\dot{H}^{\frac{1}{2}}}$$

However, the equation of Γ (1.2), the continuity of $\mathcal{R}_i$, the inequality (5.1) and Propositions 4.3 and 5.1, yields

$$\begin{aligned} \|w(t)\|_{L^2} &\leq \|\Gamma(t)\|_{L^2} + \|\mathcal{R}_1 F_2(\theta)\|_{L^2} + \|\mathcal{R}_2 F_1(\theta)\|_{L^2} \\ &\lesssim \|\Gamma(t)\|_{L^2} + \|F_2(\theta)\|_{L^2} + \|F_1(\theta)\|_{L^2} \quad (5.5) \\ &\lesssim \|\Gamma(t)\|_{L^2} + \|\theta^0\|_{L^2}. \end{aligned}$$

Therefore

$$\frac{1}{2}\frac{d}{dt}\|\Gamma(t)\|_{L^2}^2 + \|\Gamma(t)\|_{\dot{H}^{\frac{1}{2}}}^2 \leq C_0(\|\Gamma(t)\|_{L^2} + 1 + t)\|\Gamma(t)\|_{\dot{H}^{\frac{1}{2}}}.$$

The Young inequality yield

$$\frac{d}{dt}\|\Gamma(t)\|_{L^2}^2 + \|\Gamma(t)\|_{\dot{H}^{\frac{1}{2}}}^2 \leq C_0\|\Gamma(t)\|_{L^2}^2 + C_0(1+t^2).$$

Integrating this inequality, we obtain

$$\begin{aligned} \|\Gamma(t)\|_{L^2}^2 + \int_0^t \|\Gamma(\tau)\|_{\dot{H}^{\frac{1}{2}}}^2 d\tau &\leq C_0(1+t^2)e^{C_0 t} \\ &\leq \Phi_1(t). \quad (5:6) \end{aligned}$$

Now the inequality (5.5) and Young inequality implies that:

$$\|w(t)\|_{L^2}^2 \lesssim \|\Gamma(t)\|_{L^2}^2 + \|\theta^0\|_{L^2}^2.$$

Hence it follows that:

$$\|w(t)\|_{L^2}^2 + \int_0^t \|\Gamma(\tau)\|_{\dot{H}^{\frac{1}{2}}}^2 d\tau \leq \Phi_1(t).$$

This is the desired result.

Let us now prove some strong a priori estimates with regularities more stronger than previous.

5.3. Proposition :

Let (v, θ) be a solution of the system (1.1) such that $v^0 \in H^1 \cap \dot{W}^{1,p}, \theta^0 \in L^2 \cap L^\infty$ with $2 < p < \infty$ and $F \in C^1(\mathbb{R}, \mathbb{R}^2)$. Then we have for every $\gamma \in [2,4[\cap [2, p]$,

$$\|w(t)\|_{L^\gamma}^\gamma + \int_0^t \|\Gamma(\tau)\|_{L^{2\gamma}}^\gamma d\tau \leq \Phi_1(t).$$

Proof. Multiplying (1.2) by $|\Gamma|^{\gamma-2}\Gamma$ and integrating in space variable we get for every $0 < \beta < 1$, βwill be chosen later),

$$\frac{1}{\gamma}\frac{d}{dt}\|\Gamma(t)\|_{L^\gamma}^\gamma + \int_{\mathbb{R}^2} (|D|\Gamma)|\Gamma|^{\gamma-2}\Gamma \, dx \leq (\|[\mathcal{R}_1, v]F_2(\theta)\|_{\dot{H}^{1-\beta}} + \|[\mathcal{R}_2, v]F_1(\theta)\|_{\dot{H}^{1-\beta}})\||\Gamma|^{\gamma-2}\Gamma(t)\|_{\dot{H}^\beta} \tag{5.7}$$

We have from Lemma 3.3 in [15],

$$\frac{2}{\gamma}\left\||\Gamma|^{\frac{\gamma}{2}}\right\|_{\dot{H}^{\frac{1}{2}}}^2 \leq \int_{\mathbb{R}^2} (|D|\Gamma)|\Gamma|^{\frac{\gamma}{2}}\Gamma \, dx.$$

Combining this estimate with the embedding $\dot{H}^{\frac{1}{2}} \hookrightarrow L^4$, we find

$$\|\Gamma\|_{L^{2\gamma}}^\gamma \lesssim \int_{\mathbb{R}^2} (|D|\Gamma)|\Gamma|^{\frac{\gamma}{2}}\Gamma \, dx.$$

Therefore

$$(5.8)\quad \frac{1}{\gamma}\frac{d}{dt}\|\Gamma(t)\|_{L^\gamma}^\gamma + c\|\Gamma(t)\|_{L^{2\gamma}}^\gamma$$
$$\lesssim \left(\|[\mathcal{R}_1, v]F_2(\theta)\|_{\dot{H}^{1-\beta}} + \|[\mathcal{R}_2, v]F_1(\theta)\|_{\dot{H}^{1-\beta}}\right)\||\Gamma|^{\gamma-2}\Gamma(\mathrm{t})\|_{\dot{H}^\beta}.$$

Now we have from Propositions 5.1, 5.2 , the part (1) of Proposition 4.3, (5.1) and the Caldron-Zygmand estimate,

$$\|[\mathcal{R}_1, v]F_2(\theta)\|_{\dot{H}^{1-\beta}} \le \|[\mathcal{R}_1, v]F_2(\theta)\|_{H^{1-\beta}}$$
$$\lesssim \|\nabla v\|_{L^2}\|F_2(\theta)\|_{B^{-\beta}_{\infty,2}} + \|v(t)\|_{L^2}\|F_2(\theta)\|_{L^2}$$
$$\lesssim \|\mathrm{w(t)}\|_{L^2}\|F_2(\theta)\|_{L^\infty} + C_0(1+t)$$
$$\lesssim \|\mathrm{w(t)}\|_{L^2}\|\theta^0\|_{L^\infty} + C_0(1+t)$$
$$\le \Phi_1(t).$$

We have used also the embedding $L^\infty \hookrightarrow B^{-\beta}_{\infty,2}, \beta > 0$. Similarly for $\|[\mathcal{R}_2, v]F_1(\theta)\|_{\dot{H}^{1-\beta}}$. Finally we get

$$(5.9)\qquad \frac{d}{dt}\|\Gamma(t)\|_{L^\gamma}^\gamma + c\|\Gamma(t)\|_{L^{2\gamma}}^\gamma \le \Phi_1(t)\||\Gamma|^{\gamma-2}\Gamma(\mathrm{t})\|_{\dot{H}^\beta}.$$

To estimate $\||\Gamma|^{\gamma-2}\Gamma(\mathrm{t})\|_{\dot{H}^\beta}$ we use the following lemma (see [23] for a proof).

5.4. Lemma :

Let $\delta \in [2,\infty[$ and $r \in]0,1[$. Then for every smooth function η we have

$$\left\||\eta|^{\delta-2}\eta\right\|_{\dot{H}^r} \lesssim \|\eta\|_{L^{2\delta}}^{\delta-2}\|\eta\|_{\dot{H}^{r+1-\frac{2}{\delta}}}.$$

Combining this lemma with (5.9) yields

$$\frac{d}{dt}\|\Gamma(t)\|_{L^\gamma}^\gamma + c\|\Gamma(t)\|_{L^{2\gamma}}^\gamma \le \Phi_1(t)\|\Gamma\|_{L^{2\gamma}}^{\gamma-2}\|\Gamma\|_{\dot{H}^{\beta+1-\frac{2}{\gamma}}}.$$

We choose $0 < \beta < 1$ such that $\beta + 1 - \frac{2}{\gamma} = \frac{1}{2}$ which means that $\beta = \frac{2}{\gamma} - \frac{1}{2}$ this is possible for $2 \leq \gamma < 4$. Therefore

$$\frac{d}{dt}\|\Gamma(t)\|_{L^\gamma}^\gamma + c\|\Gamma(t)\|_{L^{2\gamma}}^\gamma \leq \Phi_1(t)\|\Gamma\|_{L^{2\gamma}}^{\gamma-2}\|\Gamma\|_{\dot{H}^{\frac{1}{2}}}.$$

The Young inequality

$$|ef| \leq c|e|^{\frac{\gamma}{2}} + \frac{c}{2}|f|^{\frac{\gamma}{\gamma-2}}$$

implies that

$$\frac{d}{dt}\|\Gamma(t)\|_{L^\gamma}^\gamma + c\|\Gamma(t)\|_{L^{2\gamma}}^\gamma \leq \Phi_1(t)\|\Gamma(t)\|_{\dot{H}^{\frac{1}{2}}}^{\frac{\gamma}{2}}.$$

Integrating in time with Holder inequality, we get

$$\|\Gamma(t)\|_{L^\gamma}^\gamma + c\int_0^t \|\Gamma(\tau)\|_{L^{2\gamma}}^\gamma d\tau \leq \|\Gamma^0\|_{L^\gamma}^\gamma + \Phi_1(t)\left(\int_0^t \|\Gamma(\tau)\|_{\dot{H}^{\frac{1}{2}}}^{\frac{\gamma}{2}} d\tau\right)^{\frac{\gamma}{4}}.$$

Applying Proposition 5.2, we obtain finally

$$\|\Gamma(t)\|_{L^\gamma}^\gamma + \int_0^t \|\Gamma(\tau)\|_{L^{2\gamma}}^\gamma d\tau \leq \Phi_1(t).$$

The proof of the proposition is now complete.

We need to the last following smoothing effect on Γ for the Lipschitz control of the velocity $\|\nabla v\|_{L^\infty}$.

5.5. Proposition :

Under the same assumptions of Proposition 5.3, we have for every $\gamma \in [2,4[\cap [2,p]$ and for every $\rho \in \left[1, \frac{\gamma}{2}\right[$,

$$\|\Gamma(\mathrm{t})\|_{\tilde{L}_t^\rho B_{\gamma,1}^{\frac{2}{\gamma}}} \leq \Phi_1(t).$$

Proof. Let $N \in \mathbb{N}$ to be chosen later. By definition of Besov space we have

$$\|\Gamma(t)\|_{\tilde{L}_t^\rho B_{\gamma,1}^{\frac{2}{\gamma}}} = \sum_{q<N} 2^{q\frac{2}{\gamma}} \|\Delta_q \Gamma(t)\|_{L_t^\rho L^\gamma} + \sum_{q\geq N} 2^{q\frac{2}{\gamma}} \|\Delta_q \Gamma(t)\|_{L_t^\rho L^\gamma}$$

$$:= I_1 + I_2. \quad (5.11)$$

We have, by using (5.10)

$$I_1 \leq 2^{N\frac{2}{\gamma}} \|\Gamma(t)\|_{L_t^\rho L^\gamma} \leq 2^{N\frac{2}{\gamma}} \Phi_1(t). \quad (5.12)$$

Then for I_2 we localize in frequencies the equation (1.2), we obtain

$$\begin{aligned}\partial_t \Delta_q \Gamma + \mathrm{v}.\nabla \Delta_q \Gamma + |D|\Delta_q \\ &= -[\Delta_q, v.\nabla]\Gamma + \Delta_q([\mathcal{R}_1, v.\nabla]F_2(\theta)) - \Delta_q([\mathcal{R}_2, v.\nabla]F_1(\theta)) \\ &:= h_q.\end{aligned}$$

Multiplying the above equation by $|\Delta_q \Gamma|^{\gamma-2} \Delta_q \Gamma$ and integrating in the space variable, we find

$$\frac{1}{\gamma}\frac{d}{dt}\|\Delta_q \Gamma(t)\|_{L^\gamma}^{\gamma} + \int_{\mathbb{R}^2} (|D|\Delta_q \Gamma)|\Delta_q \Gamma|^{\gamma-2}\Delta_q \Gamma dx \leq \|\Delta_q \Gamma(t)\|_{L^\gamma}^{\gamma-1} \|h_q(t)\|_{L^\gamma}.$$

Using the following generalized Bernstein inequality (see [11, 17] for a proof)

$$\forall 1 < \gamma, \ \ c2^q \|\Delta_q \Gamma(t)\|_{L^\gamma}^{\gamma} \leq \int_{\mathbb{R}^2} (|D|\Delta_q \Gamma)|\Delta_q \Gamma|^{\gamma-2}\Delta_q \Gamma dx,$$

where c depends on γ. Thus

$$\frac{1}{\gamma}\frac{d}{dt}\|\Delta_q \Gamma(t)\|_{L^\gamma}^{\gamma} + c2^q \|\Delta_q \Gamma(t)\|_{L^\gamma}^{\gamma} \leq \|\Delta_q \Gamma(t)\|_{L^\gamma}^{\gamma-1} \|h_q(t)\|_{L^\gamma}.$$

Hence it follows that

$$\|\Delta_q \Gamma(t)\|_{L^\gamma}^{\gamma-1} \frac{d}{dt}\|\Delta_q \Gamma(t)\|_{L^\gamma} + c2^q \|\Delta_q \Gamma(t)\|_{L^\gamma}^{\gamma} \leq \|\Delta_q \Gamma(t)\|_{L^\gamma}^{\gamma-1} \|h_q(t)\|_{L^\gamma}.$$

This yields

$$\frac{d}{dt}\left\|\Delta_q \Gamma(t)\right\|_{L^\gamma} + c2^q \left\|\Delta_q \Gamma(t)\right\|_{L^\gamma} \leq \left\|h_q(t)\right\|_{L^\gamma}.$$

Multiply the above inequality by e^{ct2^q} , we find

$$\frac{d}{dt}(e^{ct2^q}\left\|\Delta_q \Gamma(t)\right\|_{L^\gamma}) \leq e^{ct2^q}\left\|h_q(t)\right\|_{L^\gamma}.$$

Integrating in time, we obtain

$$\left\|\Delta_q \Gamma(t)\right\|_{L^\gamma} \leq e^{-ct2^q}\left\|\Delta_q \Gamma^0\right\|_{L^\gamma} + \int_0^t e^{-c(t-\tau)2^q}\left\|[\Delta_q, v.\nabla]\Gamma(\tau)\right\|_{L^\gamma} d\tau$$

$$+ \int_0^t e^{-c(t-\tau)2^q}\left\|\Delta_q([\mathcal{R}_1, v.\nabla]\mathrm{F}_2(\theta))(\tau)\right\|_{L^\gamma} d\tau$$

$$+ \int_0^t e^{-c(t-\tau)2^q}\left\|\Delta_q([\mathcal{R}_2, v.\nabla]\mathrm{F}_1(\theta))(\tau)\right\|_{L^\gamma} d\tau.$$

Taking the $L^\rho[0,t]$ norm, using convolution inequalities and multiplying by $2^{q\frac{2}{\gamma}}$,we obtain

$$2^{q\frac{2}{\gamma}}\left\|\Delta_q \Gamma\right\|_{L_t^\rho L^\gamma} \leq 2^{q(\frac{2}{\gamma}-\frac{1}{\rho})}\left\|\Delta_q \Gamma^0\right\|_{L^\gamma} + 2^{q(\frac{2}{\gamma}-\frac{1}{\rho})}\int_0^t \left\|[\Delta_q, v.\nabla]\Gamma(\tau)\right\|_{L^\gamma} d\tau$$

$$+2^{q(\frac{2}{\gamma}-\frac{1}{\rho})}\int_0^t \left\|\Delta_q([\mathcal{R}_1, v.\nabla]\mathrm{F}_2(\theta))(\tau)\right\|_{L^\gamma} d\tau$$

$$+ 2^{q(\frac{2}{\gamma}-\frac{1}{\rho})}\int_0^t \left\|\Delta_q([\mathcal{R}_2, v.\nabla]\mathrm{F}_1(\theta))(\tau)\right\|_{L^\gamma} d\tau. \tag{5.13}$$

For the first integral of (5.13), we use the part (1) of Lemma 2.6 and Proposition 5.3

$$\left\|[\Delta_q, v.\nabla]\Gamma\right\|_{L^\gamma} \lesssim \|\nabla v\|_{L^\gamma}\|\Gamma\|_{B^{\frac{2}{\gamma}}_{\gamma,1}} \lesssim \|\mathrm{w}\|_{L^\gamma}\|\Gamma\|_{B^{\frac{2}{\gamma}}_{\gamma,1}} \lesssim \Phi_1(t)\|\Gamma\|_{B^{\frac{2}{\gamma}}_{\gamma,1}}.$$

To estimate the second integral of (5.13), we use Propositions 5.1, 5.3, the part (2) of Proposition 4.3 and (5.1),

$$\left\|\Delta_q([\mathcal{R}_1, v.\nabla]\mathrm{F}_2(\theta))\right\|_{L^\gamma} \lesssim \|[\mathcal{R}_1, v.\nabla]\mathrm{F}_2(\theta))\|_{L^\gamma}$$
$$\lesssim \|\nabla v\|_{L^\gamma}\|\mathrm{F}_2(\theta)\|_{L^\infty} + \|\mathrm{v}\|_{L^2}\|\mathrm{F}_2(\theta)\|_{L^2}$$
$$\lesssim \|\mathrm{w}\|_{L^\gamma}\|\theta^0\|_{L^\infty} + \|\mathrm{v}\|_{L^2}\|\theta^0\|_{L^2} \le \Phi_1(t).$$

Similarly we find

$$\left\|\Delta_q([\mathcal{R}_1, v.\nabla]\mathrm{F}_2(\theta))\right\|_{L^\gamma} \le \Phi_1(t).$$

Summing the inequality (5.13) on $q \ge N$,

$$\sum_{q\ge N} 2^{q\frac{2}{\gamma}}\|\Delta_q\Gamma\|_{L^\rho_t L^\gamma} \lesssim \sum_{q\ge N} 2^{q(\frac{2}{\gamma}-\frac{1}{\rho})}\left\|\Delta_q\Gamma^0\right\|_{L^\gamma} + \sum_{q\ge N} 2^{q(\frac{2}{\gamma}-\frac{1}{\rho})}\int_0^t \Phi_1(\tau)\|\Gamma(\tau)\|_{B^{\frac{2}{\gamma}}_{\gamma,1}} d\tau$$
$$+ \sum_{q\ge N} 2^{q(\frac{2}{\gamma}-\frac{1}{\rho})}\int_0^t \Phi_1(\tau)d\tau$$

Therefore,

$$\sum_{q\ge N} 2^{q\frac{2}{\gamma}}\left\|\Delta_q\Gamma\right\|_{L^\rho_t L^\gamma} \lesssim 2^{-N\left(\frac{1}{\rho}-\frac{2}{\gamma}\right)}\|\Gamma^0\|_{L^\gamma} + 2^{-N\left(\frac{1}{\rho}-\frac{2}{\gamma}\right)}\Phi_1(t)\|\Gamma\|_{L^1_t B^{\frac{2}{\gamma}}_{\gamma,1}}$$
$$+ 2^{-N\left(\frac{1}{\rho}-\frac{2}{\gamma}\right)}\Phi_1(t).$$

Since $\|\Gamma^0\|_{L^\gamma} \lesssim \|w^0\|_{L^\gamma} + \|\theta^0\|_{L^\gamma}$, then

$$\sum_{q\ge N} 2^{q\frac{2}{\gamma}}\left\|\Delta_q\Gamma\right\|_{L^\rho_t L^\gamma} \le 2^{-N\left(\frac{1}{\rho}-\frac{2}{\gamma}\right)}\Phi_1(t)\|\Gamma\|_{L^1_t B^{\frac{2}{\gamma}}_{\gamma,1}} \le 2^{-N\left(\frac{1}{\rho}-\frac{2}{\gamma}\right)}\Phi_1(t)t^{1-\frac{1}{\rho}}\|\Gamma\|_{L^\rho_t B^{\frac{2}{\gamma}}_{\gamma,1}}$$
$$\le 2^{-N\left(\frac{1}{\rho}-\frac{2}{\gamma}\right)}\Phi_1(t)\|\Gamma\|_{\tilde{L}^\rho_t B^{\frac{2}{\gamma}}_{\gamma,1}}. \quad (5.14)$$

Putting (5.12) and (5.14) into (5.11), we obtain

$$\|\Gamma\|_{\tilde{L}_t^\rho B_{\gamma,1}^{\frac{2}{\gamma}}} \leq 2^{N\frac{2}{\gamma}}\Phi_1(t) + 2^{-N\left(\frac{1}{\rho}-\frac{2}{\gamma}\right)}\Phi_1(t)\|\Gamma\|_{\tilde{L}_t^\rho B_{\gamma,1}^{\frac{2}{\gamma}}}.$$

Finally we choose N such that

$$2^{-N\left(\frac{1}{\rho}-\frac{2}{\gamma}\right)}\Phi_1(t) \approx \frac{1}{2}.$$

This implies that

$$\|\Gamma\|_{\tilde{L}_t^\rho B_{\gamma,1}^{\frac{2}{\gamma}}} \leq \Phi_1(t).$$

This is the desired result.

The aim now of the following proposition is to get estimates on $\|\nabla v\|_{L^\infty}$.

5.6. Proposition :

Assume that (v,θ) be a smooth solution of the system (1.1) with $F \in C^2(\mathbb{R},\mathbb{R}^2)$ Let $v^0 \in H^1 \cap \dot{W}^{1,p}, 2 < p < \infty$ and $\theta^0 \in L^2 \cap B_{\infty,1}^0$. Then we have $\forall\rho \in [1,2[\cap \left[1,\frac{p}{2}\right[$,

$$\|w\|_{\tilde{L}_t^\rho B_{\infty,1}^0} + \|v\|_{\tilde{L}_t^\rho B_{\infty,1}^1} + \|\theta(t)\|_{\tilde{L}_t^\infty B_{\infty,1}^0} \leq \Phi_1(t).$$

Proof. First from the embedding $B_{\gamma,1}^{\frac{2}{\gamma}} \hookrightarrow B_{\infty,1}^0$, we immediately get from the Proposition 5.5 that for $t \in \mathbb{R}_+$

$$\|\Gamma(t)\|_{\tilde{L}_t^\rho B_{\infty,1}^0} \leq \Phi_1(t). \tag{5.15}$$

Using (5.15) for $\rho = 1$,

$$\|w\|_{L_t^1 B_{\infty,1}^0} \leq \|\Gamma\|_{L_t^1 B_{\infty,1}^0} + \int_0^t \|\mathcal{R}_1 F_2(\theta(\tau))\|_{B_{\infty,1}^0} d\tau + \int_0^t \|\mathcal{R}_2 F_1(\theta(\tau))\|_{B_{\infty,1}^0} d\tau$$

$$\leq \Phi_1(t) + \int_0^t \|\mathcal{R}_1 F_2(\theta(\tau))\|_{B_{\infty,1}^0} d\tau + \int_0^t \|\mathcal{R}_2 F_1(\theta(\tau))\|_{B_{\infty,1}^0} d\tau \tag{5.16}$$

Now from Bernstein inequality, Propositions 4.1, 5.1 and (5.1), we find,

$$\|\mathcal{R}_1 \mathrm{F}_2(\theta)\|_{B^0_{\infty,1}} = \|\Delta_{-1}\mathcal{R}_1 \mathrm{F}_2(\theta)\|_{L^\infty} + \sum_{q\in\mathbb{N}} \|\Delta_q \mathcal{R}_1 \mathrm{F}_2(\theta)\|_{L^\infty}$$

$$\leq \|\Delta_{-1}\mathcal{R}_1 \mathrm{F}_2(\theta)\|_{L^2} + \|\mathrm{F}_2(\theta)\|_{B^0_{\infty,1}}$$

$$(5.17) \qquad \lesssim \|\theta^0\|_{L^2} + \|\mathrm{F}_2(\theta)\|_{\tilde{L}^\infty_t B^0_{\infty,1}},$$

where we have used the fact that $\Delta_q \mathcal{R}_i$ is uniformly bounded (with respect to $q \in \mathbb{N}$ in L^p for every $1 \leq p \leq \infty$. Similarly, we find

$$(5.18) \qquad \|\mathcal{R}_2 \mathrm{F}_1(\theta)\|_{B^0_{\infty,1}} \lesssim \|\theta^0\|_{L^2} + \|\mathrm{F}_1(\theta)\|_{\tilde{L}^\infty_t B^0_{\infty,1}}.$$

It remains then to estimate $\|\mathrm{F}_i(\theta)\|_{\tilde{L}^\infty_t B^0_{\infty,1}}, \forall i = 1,2$. For this purpose, we use the following theorem see [17] for a proof

5.7. Theorem :

Let $F \in C^{[s]+2}, s$ a positive real number and F vanishing at 0. If u belongs to $B^s_{p,r} \cap L^\infty$, with $(p,r) \in [1,+\infty]^2$, then Fou belongs to $B^s_{p,r}$ and we have

$$\|Fou\|_{B^s_{p,r}} \leq C_s sup_{|x|\leq C\|u\|_{L^\infty}} \left\|F^{[s]+2}(x)\right\|_{L^\infty} \|u\|_{B^s_{p,r}}.$$

Since $F \in C^2(\mathbb{R},\mathbb{R}^2)$, we deduce that

$$\|F_i(\theta)\|_{B^0_{\infty,1}} \leq \|\theta^0\|_{B^0_{\infty,1}}.$$

Since $(\partial_t + v.\nabla)F_i(\theta) = 0$ then applying Proposition 3.2, we find

$$(5.19) \qquad \|F_i(\theta)\|_{\tilde{L}^\infty_t B^0_{\infty,1}} \leq \|F_i(\theta^0)\|_{B^0_{\infty,1}}(1 + \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau)$$

Therefore

$$\|w\|_{L_t^1 B_{\infty,1}^0} \leq \Phi_1(t) + \left(\|\theta^0\|_{L^2} + \|F_2(\theta^0)\|_{B_{\infty,1}^0} + \|F_1(\theta^0)\|_{B_{\infty,1}^0}\right) t$$
$$+ (\|F_2(\theta^0)\|_{B_{\infty,1}^0} + \|F_1(\theta^0)\|_{B_{\infty,1}^0}) \int_0^t \|\nabla v\|_{L_\tau^1 L^\infty} d\tau$$
$$\leq \Phi_1(t) + C_0 t + C_0 \int_0^t \|v\|_{L_\tau^1 B_{\infty,1}^1} d\tau$$
$$\leq \Phi_1(t) + C_0 \int_0^t \|v\|_{L_\tau^1 B_{\infty,1}^1} d\tau, \tag{5.20}$$

we have used the embedding $B_{\infty,1}^1 \hookrightarrow Lip$. It remains then to estimate $\|v\|_{L_t^1 B_{\infty,1}^1}$. For this purpose, we use the definition of Besov space, Bernstein inequality and Proposition 5.1, we have for $j \geq -1$,

$$\|v\|_{L_t^1 B_{\infty,1}^1} \leq C\|\Delta_{-1} v\|_{L_t^1 L^\infty} + \sum_{j\in\mathbb{N}} 2^j \|\Delta_j v\|_{L_t^1 L^\infty}$$
$$\lesssim \|v\|_{L_t^1 L^2} + \|w\|_{L_t^1 B_{\infty,1}^0}$$
$$\lesssim C_0(1+t^2) + \|w\|_{L_t^1 B_{\infty,1}^0}, \tag{5.21}$$

Where we have used the classical fact $\|\Delta_j v\|_{L^p} \approx 2^{-j} \|\Delta_j z\|_{L^p}$ uniformly in j for every $j \in [1,\infty]$.

Putting (5.21) into (5.20) and using Gronwall inequality, we immediately get

$$\|w\|_{L_t^1 B_{\infty,1}^0} \leq \Phi_1(t) + C_0 \int_0^t (1+\tau^2)\, d\tau + C_0 \int_0^t \|w\|_{L_\tau^1 B_{\infty,1}^1}\, d\tau$$
$$\leq \Phi_1(t). \tag{5.22}$$

Hence, we immediately get

$$\|v\|_{L_t^1 B_{\infty,1}^1} \leq \Phi_1(t). \tag{5.23}$$

By the Besov embedding $B_{\infty,1}^1 \hookrightarrow Lip$, we find

$$\|\nabla v\|_{L_t^1 L^\infty} \leq \Phi_1(t). \tag{5.24}$$

Now from (5.19) and (5.24), we obtain

$$(5.25) \qquad \sum_{i=1}^{2} \|F_i(\theta)\|_{\tilde{L}_t^\infty B^0_{\infty,1}} \leq \Phi_1(t).$$

This leads in (5.17) and (5.18), to

$$(5.26) \qquad \|\mathcal{R}_1 F_2(\theta)\|_{\tilde{L}_t^\infty B^0_{\infty,1}} + \|\mathcal{R}_2 F_1(\theta)\|_{\tilde{L}_t^\infty B^0_{\infty,1}} \leq \Phi_1(t).$$

Now we can applying Proposition 3.2 to the equation θ to get

$$\|\theta\|_{\tilde{L}_t^\infty B^0_{\infty,1}} \leq \|\theta^0\|_{B^0_{\infty,1}} \left(1 + \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\right) \leq \Phi_1(t),$$

we have used the inequality (5.24). Finally (5.15),(5.26) and Holder inequality yields for every $1 \leq \rho < \frac{p}{2}$,

$$\begin{aligned} &\|\mathrm{w}\|_{\tilde{L}_t^\rho B^0_{\infty,1}} \\ &\leq \|\Gamma\|_{\tilde{L}_t^\rho B^0_{\infty,1}} + \|\mathcal{R}_1 F_2(\theta)\|_{\tilde{L}_t^\rho B^0_{\infty,1}} + \|\mathcal{R}_2 F_1(\theta)\|_{\tilde{L}_t^\rho B^0_{\infty,1}} \\ &\leq \Phi_1(t) + t^{\frac{1}{\rho}}(\|\mathcal{R}_1 F_2(\theta)\|_{\|\mathrm{w}\|_{\tilde{L}_t^\infty B^0_{\infty,1}} B^0_{\infty,1}} + \|\mathcal{R}_2 F_1(\theta)\|_{\tilde{L}_t^\infty B^0_{\infty,1}}) \\ (5.27) \qquad &\leq \Phi_1(t). \end{aligned}$$

Using now Proposition 5.1, (5.27) and Bernstein inequality,

$$\begin{aligned} \|v\|_{\tilde{L}_t^\rho B^1_{\infty,1}} &\leq \|\Delta_{-1} v\|_{L_t^\rho L^\infty} + \|\mathrm{w}\|_{\tilde{L}_t^\rho B^0_{\infty,1}} \\ &\lesssim \|v\|_{L_t^\rho L^2} + \|\mathrm{w}\|_{\tilde{L}_t^\rho B^0_{\infty,1}} \leq \Phi_1(t). \end{aligned}$$

The proof of the proposition is now complete.

Finally we prove the L^p norm of the vorticity $\forall p \geq 4$.

5.8. Proposition :

Under the hypotheses of Proposition 5.6, we have for every $t \in \mathbb{R}_+$,

$$\|w(t)\|_{L^p} \leq \Phi_1(t).$$

Proof. Recall that $\Gamma = w - \mathcal{R}_1 F_2(\theta) + \mathcal{R}_2 F_1(\theta)$ satisfies the equation

$$\partial_t \Gamma + v.\nabla\Gamma + |D|\Gamma = [\mathcal{R}_1, v.\nabla]F_2(\theta) - [\mathcal{R}_2, v.\nabla]F_1(\theta)$$

Using Proposition 3.2 we find

$$\|\Gamma(t)\|_{L^p} \leq \|\Gamma^0\|_{L^p} + \int_0^t \|[\mathcal{R}_1, v.\nabla]F_2(\theta)(\tau)\|_{L^p}\, d\tau + \int_0^t \|[\mathcal{R}_2, v.\nabla]F_1(\theta)(\tau)\|_{L^p}\, d\tau.$$

We use the embedding $B^0_{p,1} \hookrightarrow L^p$, the part (3) of Proposition 4.3 with $F(\theta)$, Proposition 5.1, (5.1) and (5.25), we obtain $\forall 2 \leq p < \infty$,

$$\begin{aligned}\|[\mathcal{R}_1, v.\nabla]F_2(\theta)\|_{L^p} &\lesssim \|[\mathcal{R}_1, v.\nabla]F_2(\theta)\|_{B^0_{p,1}} \\ &\lesssim \|\nabla v\|_{L^p}(\|F_2(\theta)\|_{B^0_{\infty,1}} + \|F_2(\theta)\|_{L^p}) \\ &\leq \|\mathrm{w}\|_{L^p}(\Phi_1(t) + \|\theta^0\|_{L^p}) \\ &\leq \Phi_1(t)\|\mathrm{w}\|_{L^p}.\end{aligned}$$

Similarly for $\|[\mathcal{R}_2, v.\nabla]F_1(\theta)\|_{L^p}$ we find

$$\|[\mathcal{R}_2, v.\nabla]F_1(\theta)\|_{L^p} \leq \Phi_1(t)\|\mathrm{w}\|_{L^p}.$$

Therefore

$$\|\Gamma(t)\|_{L^p} \leq \|\Gamma^0\|_{L^p} + \int_0^t \Phi_1(\tau)\|\mathrm{w}(\tau)\|_{L^p}\, d\tau.$$

On the other hand we have

$$\begin{aligned}\|\mathrm{w}(t)\|_{L^p} \leq\ & \|\Gamma(t)\|_{L^p} + \|\mathcal{R}_1 F_2(\theta(t))\|_{L^p} + \|\mathcal{R}_2 F_1(\theta(t))\|_{L^p} \\ \leq\ & \|\Gamma(t)\|_{L^p} + \|F_2(\theta(t))\|_{L^p} + \|F_1(\theta(t))\|_{L^p} \\ \leq\ & \|\Gamma(t)\|_{L^p} + \|\theta^0\|_{L^p}.\end{aligned}$$

We have used the continuity of Riesz transform, the inequality (5.1) and Proposition 5.1. Hence

$$\|\mathrm{w}(t)\|_{L^p} \leq \|\Gamma^0\|_{L^p} + \|\theta^0\|_{L^p} + \int_0^t \Phi_1(\tau)\|\mathrm{w}(\tau)\|_{L^p}\, d\tau$$

$$\lesssim \|w^0\|_{L^p} + \|\theta^0\|_{L^p} + \int_0^t \Phi_1(\tau)\|w(\tau)\|_{L^p}\, d\tau.$$

Gronwall's inequality gives

$$\|w(t)\|_{L^p} \leq \Phi_2(t).$$

This proves the proposition.

5.2. Uniqueness.

We will prove a uniqueness result of the system (1.1) in the following

Space

$$\mathcal{A}_T := L_T^\infty H^1 \cap L_T^1 B_{\infty,1}^1 \times L_T^\infty\left(L^2 \cap B_{\infty,1}^0\right).$$

Let $\{v_j, \theta_j\}, j = 1,2$ two solutions of the system (1.1) with initial data $(v_j^0, \theta_j^0), j = 1,2$ belonging to the space $\mathcal{A}_{T.}$ We set

$$v = v_1 - v_2, \quad \theta = \theta_1 - \theta_2, \; p = p_1 - p_2 \quad and \quad G(\theta) = F(\theta_1) - F(\theta_2).$$

Then we find the equations

$$\begin{cases} \partial_t v + v_2.\nabla v + |D|v + \nabla p = -v.\nabla v_1 + G(\theta) \\ \partial_t \theta + v_2.\nabla\theta = -v.\nabla\theta_1 \\ v(t=0) = v^0, \qquad \theta(t = 0 = \theta^0\,. \end{cases}$$

To estimate v, we can write then $v = \widetilde{v_1} + \widetilde{v_2}$ where $\widetilde{v_1}$ and $\widetilde{v_2}$ solve respectively the following equations

$$\partial_t\widetilde{v_1} + v_2.\nabla\widetilde{v_1} + |D|\widetilde{v_1} + \nabla p_1 = -v.\nabla v_1$$

$$\partial_t\widetilde{v_2} + v_2.\nabla\widetilde{v_2} + |D|\widetilde{v_2} + \nabla p_2 = G(\theta).$$

To estimate $\widetilde{v_1}$, we use Proposition 3.3 for $\rho = 1$ and $s = 0$, then for $\widetilde{v_2}$, ; we use Proposition 3.3 for $\rho = \infty$ and $s = 0$. Thus for every $0 \leq t \leq T$,;

$$\|v\|_{\widetilde{L_t^\infty} B^0_{2,\infty}} \leq \|\widetilde{v_1}\|_{\widetilde{L_t^\infty} B^0_{2,\infty}} + \|\widetilde{v_2}\|_{\widetilde{L_t^\infty} B^0_{2,\infty}}.$$

Therefore

$$(5.28) \qquad \|v\|_{\widetilde{L_t^\infty} B^0_{2,\infty}} \leq e^{CV_2(t)}\left(\|v^0\|_{B^0_{2,\infty}} + \|v.\nabla v_1\|_{\widetilde{L_t^1} B^0_{2,\infty}} + (1+t)\|G(\theta)\|_{\widetilde{L_t^\infty} B^{-1}_{2,\infty}}\right),$$

with $V_2(t) := \int_0^t \|\nabla v_2(\tau)\|_{L^\infty}\, d\tau$. Using Lemma 2.6-(2), we obtain

$$\|v.\nabla v_1\|_{B^0_{2,\infty}} \lesssim \|v\|_{L^2}\|v_1\|_{B^1_{\infty,1}}.$$

To estimate $\|v\|_{L^2}$ we use the following Lemma (see [23] for a proof).

5.9. Lemma :

Let $v \in H^1$ then we have

$$\|v\|_{L^2} \lesssim \|v\|_{B^0_{2,\infty}} \log\left(e + \frac{\|v\|_{H^1}}{\|v\|_{B^0_{2,\infty}}}\right).$$

Hence it follows that

$$\|v.\nabla v_1\|_{B^0_{2,\infty}} \lesssim \|v_1\|_{B^1_{\infty,1}}\|v\|_{B^0_{2,\infty}} log\left(e + \frac{\|v\|_{H^1}}{\|v\|_{B^0_{2,\infty}}}\right)$$

$$\lesssim \|v_1\|_{B^1_{\infty,1}}\|v\|_{B^0_{2,\infty}} log\left(e + \frac{1}{\|v\|_{B^0_{2,\infty}}}\right)\log(e + \|v\|_{H^1})$$

$$(5.29) \qquad \lesssim \|v_1\|_{B^1_{\infty,1}} \log(e + \|v\|_{H^1})\, \mu\left(\|v\|_{B^0_{2,\infty}}\right),$$

where $\mu(x) = xlog(e + \frac{1}{x})$. Putting (5.29) into (5.28), we find

$$\|v\|_{\widetilde{L_t^\infty} B^0_{2,\infty}} \leq e^{CV_2(t)}\left(\|v^0\|_{B^0_{2,\infty}} + \int_0^t \|v_1\|_{B^1_{\infty,1}} \log(e + \|v\|_{H^1})\, \mu\left(\|v\|_{B^0_{2,\infty}}\right) d\tau\right)$$

$$(5.30) \qquad + e^{CV_2(t)}(1+t)\|G(\theta)\|_{\widetilde{L_t^\infty} B^{-1}_{2,\infty}}.$$

To estimate $\|G(\theta)\|_{\widetilde{L}_t^\infty B_{2,\infty}^{-1}}$ with $G(\theta) = F(\theta_1) - F(\theta_2)$. we use the equations

$$\partial_t \theta_j + v_j . \nabla \theta_j = 0, \quad j = 1,2.$$

Multiplying the above equation by $\grave{F}(\theta_j)$, we find

$$\partial_t F(\theta_j) + v_j . \nabla \mathrm{F}(\theta_j) = 0.$$

Then we immediately have for $G(\theta) = F(\theta_1) - F(\theta_2)$, that

$$\partial_t G(\theta) + v_2 . \nabla G(\theta) = -v . \nabla F(\theta_1),$$

applying then Proposition 3.1 with $p = 2$ yields

$$(5.31) \qquad \|G(\theta)\|_{\widetilde{L}_t^\infty B_{2,\infty}^{-1}} \lesssim e^{C\|v_2\|_{L_t^1 B_{\infty,1}^1}} \left(\|G(\theta^0)\|_{B_{2,\infty}^{-1}} + \int_0^t \|v . \nabla F(\theta_1(\tau))\|_{B_{2,\infty}^{-1}} \, d\tau \right).$$

As before, using Lemma 2.6-(2) and Lemma 5.9, yield

$$\|v . \nabla F(\theta_1)\|_{B_{2,\infty}^{-1}} \lesssim \|v\|_{L^2} \|F(\theta_1)\|_{B_{\infty,1}^0}$$

$$\lesssim \|v\|_{B_{2,\infty}^0} \log\left(e + \frac{1}{\|v\|_{B_{2,\infty}^0}} \right) \log(e + \|v\|_{H^1}) \|F(\theta_1)\|_{B_{\infty,1}^0}$$

$$\lesssim \|F(\theta_1)\|_{B_{\infty,1}^0} \log(e + \|v\|_{H^1}) \, \mu\left(\|v\|_{B_{2,\infty}^0} \right).$$

Thus

$$\|G(\theta)\|_{\widetilde{L}_t^\infty B_{2,\infty}^{-1}} \lesssim e^{C\|v_2\|_{L_t^1 B_{\infty,1}^1}} \Bigg(\|G(\theta^0)\|_{B_{2,\infty}^{-1}}$$

$$+ \int_0^t \|F(\theta_1)\|_{B_{\infty,1}^0} \log(e + \|v\|_{H^1}) \, \mu\left(\|v\|_{B_{2,\infty}^0} \right) d\tau \Bigg).$$

Using now Theorem 5.7, we get

(5.32) $\|G(\theta)\|_{\widetilde{L_t^\infty} B_{2,\infty}^{-1}}$

$$\lesssim e^{C\|v_2\|_{L_t^1 B_{\infty,1}^1}} \left(\|G(\theta^0)\|_{B_{2,\infty}^{-1}} + \int_0^t \|\theta_1\|_{B_{\infty,1}^0} \log(e + \|v\|_{H^1})\, \mu\left(\|v\|_{B_{2,\infty}^0}\right) d\tau \right).$$

Therefore, it follows from (5.30) and (5.32),

$$\|v\|_{\widetilde{L_t^\infty} B_{2,\infty}^0} + \|G(\theta)\|_{\widetilde{L_t^\infty} B_{2,\infty}^{-1}} \lesssim e^{C\|v_2\|_{L_t^1 B_{\infty,1}^1}} \left(\|v^0\|_{B_{2,\infty}^0} + \|G(\theta^0)\|_{B_{2,\infty}^{-1}} \right)$$
$$+ e^{C\|v_2\|_{L_t^1 B_{\infty,1}^1}} \int_0^t (\|v_1(\tau)\|_{B_{\infty,1}^1} + \|\theta_1(\tau)\|_{B_{\infty,1}^1})$$
$$\log(e + \|v\|_{H^1})\, \mu\left(\|v\|_{B_{2,\infty}^0}\right) d\tau. \quad (5.33)$$

If we set $Y(t) = \|v\|_{\widetilde{L_t^\infty} B_{2,\infty}^0} + \|G(\theta)\|_{\widetilde{L_t^\infty} B_{2,\infty}^{-1}}$, then we find

$$Y(t) \leq g(t) \left(Y(0) + \int_0^t (\|v_1(\tau)\|_{B_{\infty,1}^1} + \|\theta_1(\tau)\|_{B_{\infty,1}^0} \right)$$
$$\log(e + \|v\|_{H^1})\, \mu\big(Y(\tau)\big) d\tau,$$

with g a function depending on $\|(v_j, \theta_j)\|_{\mathcal{A}_T}$ and on the variable times. The key lemma is the following, known as the Osgood Lemma (for a proof see [6]).

5.10. Lemma :

Let ρ be a measurable, positive function, γ a positive, locally integrable function and μ a continuous, increasing function. Assume that for a positive real number c, the function ρ satisfies

$$0 \le \rho(t) \le c + \int_0^t \gamma(\tau)\mu\big(\rho(\tau)\big)\, d\tau, \qquad \forall t \in \mathbb{R}_+.$$

If c different from zero, then we have

$$-\mathcal{M}\big(\rho(t)\big) + \mathcal{M}(c) \le \int_0^t \gamma(\tau) d\tau,$$

where

$$\mathcal{M}(x) := \int_x^1 \frac{dr}{\mu(r)}.$$

If $c = 0$ and if μ satisfies $\int_0^1 \frac{dr}{\mu(r)} = +\infty$, then the function ρ is identically zero.

Applying this lemma, with $c = g(t)Y(0) \neq 0$, then

$$-\mathcal{M}\big(Y(t)\big) + \mathcal{M}(g(t)Y(0))$$
$$\le \int_0^t (\|v_1(\tau)\|_{B^1_{\infty,1}} + \|\theta_1(\tau)\|_{B^0_{\infty,1}}) \log(e + \|v(\tau)\|_{H^1})\, d\tau,$$

where we have used Lemma 5.2.1 in [6] for $\mu(r) = r(1 - \log r)$.Taking a double exponential with simple calculations , we get the uniqueness.

5.3. Existence.

According to Lemma 4.4 in [17], we can first smooth our initial data $v_n(0) = v_{n,0}$ and $\theta_n(0) = \theta_{n,0}$. We consider then the following system,

$$(5.34) \qquad \begin{cases} \partial_t v_n + v_n.\nabla v_n + |D| v_n + \nabla p_n = F(\theta_n) \\ \partial_t \theta_n + v_n.\nabla \theta_n = 0, \qquad div\, v_n = 0 \\ v_n(t=0) = v_{n,0}, \qquad \theta_n(t=0) = \theta_{n,0}. \end{cases}$$

The proof can be divided into two steps. The first step establish the local well-posedness of the system (5.34). The second step veries for some $0 < T$ that v_n is a Cauchy sequence in the space $L_T^\infty B_{2,\infty}^0$.

Step 1. A similarly argument as in [21], we can prove the local well-posedness of the system (5.34). The global existence of these solutions is governed by $V_n(t) = \|\nabla v_n\|_{L_T^1 L^\infty}$ which is finite by (5.24). Now the Lipschitz norm of the velocity can not blow up in finite time by Proposition 5.6, then the solution (v_n, θ_n) is globally defined. Once again from the a priori estimates we have for $1 \leq \rho < \frac{p}{2}$,

$$\|v_n\|_{L_T^\infty(H^1 \cap \dot{W}^{1,p})} + \|v_n\|_{L_T^\rho B_{\infty,1}^1} \leq \Phi_2(T)$$

and

$$\|\theta_n\|_{L_T^\infty(L^2 \cap B_{\infty,1}^0)} \leq \Phi_2(T).$$

Then there exist (v, θ) satisfying the above estimates such that (v_n, θ_n) weakly convergent to (v, θ) up to the extraction of a subsequence.

Step 2. To show that v_n is a Cauchy sequence in the space $L_T^\infty B_{2,\infty}^0$, we consider the difference $v_{n,m} = v_n - v_m$ and $\theta_{n,m} = \theta_n - \theta_m$.
Now, if we have

$$d_{n,m} := \|v_{n,0} - v_{m,0}\|_{B_{2,\infty}^0} + \|\theta_{n,0} - \theta_{m,0}\|_{B_{2,\infty}^{-1}} \leq a(T),$$

then we have,

$$\|v_{n,m}\|_{L_T^\infty B_{2,\infty}^0} + \|\theta_{n,m}\|_{L_T^\infty B_{2,\infty}^{-1}} \leq b(T)\left(d_{n,m}\right)^{c(T)}.$$

This proves that v_n is of Cauchy and hence it converges strongly to v in $L_T^\infty B_{2,\infty}^0$. By interpolation we obtain the strong convergence of v_n to v. Thus v_n strongly converges by Cauchy Schartz in $L_T^2(\mathbb{R}^2)$. Thus $v_n \times v_n$ strongly converges by Cauchy Schartz in $L_T^2(\mathbb{R}^2)$. Now Proposition 5.1 implies the weakly convergence of θ_n to θ in $L_T^2(\mathbb{R}^2)$, we have then $v_n\theta_n$

converge weakly to $v\theta$. It then suffices to pass to the limit in (5.34) and finally get that (v, θ) is a solution of our system (1.1).

Future work : In the future work, we plan to investigate how to proof the global well$-$posedness result for (1.1), but the second equation of (1.1) with a dissipation term, that is $\partial_t\theta + v.\nabla\theta - \Delta\theta = 0$.

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