الحلول التقريبية للمعادلات التفاضلية الجزئية من الرتبة الكسرية

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الملخص العربي :

المعادلات التفاضلية الجزئية من الرتبة الكسرية كثيرا ما تظهر في مختلف المجالات العلمية والهندسية . وبالتالي زاد الاحتياج لحل هذه المعادلات . معظم هذه المعادلات يكون من الصعب أو من المستحيل حلها تحليليا ؛ لذلك نحتاج إلى طرق عددية وتقريبية فعالة ودقيقة لحلها . أصبحت من البحوث المأخوذة في الاعتبار.

في هذه الورقة البحثية استخدمت طريقة التشويش المضطرب لحل بعض أنواع من المعادلات التفاضلية الجزئية من الرتبة الكسرية. تم الحصول على الحل المضبوط أو الحل التقريبي لهذه المعادلات. الأمثلة المقترحة أثبتت دقة وسهولة تطبيق هذه الطريقة .

Approximate Solutions for Fractional Partial Differential Equations

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Abstract:

Fractional partial differential equations appear more and more frequently in various branches of science and engineering. There is a growing need to find the solution of these equations. However, most of these equations are difficult or impossible to solve analytically. So, finding accurate and efficient numerical and approximate methods for solving partial differential equations of fractional order has become needed and active research undertaking.

In the present paper, Modified of homotopy perturbation method proposed by He has been used to obtain the solution of some types of fractional partial differential equations with variable coefficients. Exact and/or approximate analytical solutions of these equations are obtained. The fractional derivative is taken in Caputo sense. We make the tables to compare between the approximate solutions and the exact solutions for fractional partial differential equations.

1. Introduction

Fractional differential equations have attracted much attention recently, see for instance [14],[20],[21].[23],[32]. This is mostly due to the fact that fractional calculus provides an efficient and excellent instrument for description of many particular dynamical phenomena arising in engineering and scientific disciplines such as, physics, chemical, biology, electrochemistry, electromagnetic, control, porous media and many more, see for instance, [2],[22],[33]. Fractional partial differential equations have received considerable interest in recent years and have extensively investigated and applied for many problems which are modeled in various areas for instance [25],[26],[29],[34]. Several methods for solving fractional partial differential equations have been presented in [1],[3],[4],[5-13],[15-19],[24],[27],[28],[30],[31],[35-37].

In present paper, we will consider the fractional partial differential equations of the form:

$$\left.\begin{array}{l}\dfrac{\partial^2 u}{\partial t^2}=f\ (x,y,z)u_{xx}+g\ (x,y,z)u_{yy}+h\ (x,y,z)u_{zz}\\ 0<x<a,0<x<b,0<x<c,t>0\end{array}\right\} \qquad (1.1)$$

Subject to the boundary conditions:

$$\left.\begin{array}{l} u_x\left(0,y,z,t\right)=f_1\left(y,z,t\right)\mu_x\left(a,y,z,t\right)=f_2\left(y,z,t\right), \\ u_y\left(x,0,z,t\right)=g_1\left(x,z,t\right)\mu_y\left(x,b,z,t\right)=g_2\left(x,z,t\right), \\ u_z\left(x,y,0,t\right)=h_1\left(x,y,t\right)\mu_z\left(x,y,c,t\right)=h_2\left(x,y,t\right) \end{array}\right\} \quad (1.2)$$

and initial conditions

$$u\left(x,y,z,0\right)=\psi\left(x,y,z\right)\mu_x\left(x,y,z,0\right)=\varphi\left(x,y,z\right) \quad (1.3)$$

Where α is a parameter describing the fractional derivative.In the case of $0<\alpha\le 1$ the Eq.(1.1) reduce to the fractional heat-like equation with variable coefficients and in the case $1<\alpha\le 2$ the Eq.(1.1) reduce to the fractional wave-like equation with variable coefficients.

In [24] Mohyud-din et.al solving this type of equations by using homotopy analysis method. On the other hand, [31] applied HPM to solve Quadratic Ricccati differential equation of fractional order, also Momani and Odibat using this method for solving nonlinear partial differential equations of fractional order.

Our aim is apply HPM to solve fractional partial differential equations of the form (1.1). The modified homotopy perturbation method [31] will be adopted.

2. Preliminaries

In this section, we give some basic definitions and properties of fractional calculus theory which used in this paper.

Definition 2.1:

A real function $f(x), x>0$ is said to be in space $C_\mu, \mu\in\mathbb{R}$ if there exists a real number $p>\mu$,such that

$f(x)=x^p f_1(x)$ where $f_1(x)\in C(0,\infty)$, and it is said to be in the space C_μ^n if $f^n \in \mathrm{R}_\mu, n\in \mathrm{N}$.

Definition 2.2:

The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu, \mu \geq -1$ is defined as:

$$J^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt, \qquad \alpha>0, \quad t>0 \qquad (2.1)$$

In particular $J^0 f(x)=f(x)$

For $\beta \geq 0$ and $\gamma \geq -1$, some properties of operator J^α

1. $J^\alpha J^\beta f(x)=J^{\alpha+\beta} f(x)$
2. $J^\alpha J^\beta f(x)=J^\beta J^\alpha f(x)$
3. $J^\alpha x^\gamma=\dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$

Definition 2.3:

The Caputo fractional derivative of $f \in C_{-1}^m$, $m\in \mathrm{N}$ is defined as:

$$D^\alpha f(x)=\frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1} f^m(t)dt, \qquad m-1<\alpha\leq m \qquad (2.$$

Lemma 2.1:

If $m-1<\alpha\leq m$, $m\in \mathrm{N}$, $f\in C_\mu^m$, $\mu>-1$ then the following two properties hold

1. $D^\alpha\left[J^\alpha f(x)\right]=f(x)$
2. $J^\alpha\left[D^\alpha f(x)\right]=f(x)-\sum_{k=1}^{m-1} f^k(0)\dfrac{x^k}{k!}$.

3. Homotopy Perturbation Method

The homotopy perturbation method was first proposed by He [6], [11] is applied to various problems, see [5],[12]. To illustrate the basic idea of this method, we consider the following nonlinear differential equation

$$D^{\alpha}u(x)+L(u(x))+N(u(x))=f(t), \quad t>0,\ m-1<\alpha\le m \qquad (3.1)$$

with boundary conditions

$$B\left(u,\frac{\partial u}{\partial n}\right)=g(x,y,t), \qquad r\in\Gamma \qquad (3.2)$$

Subject to initial condition:

$$u^{(k)}(0)=c_k \qquad (3.3)$$

where $D^{\alpha}u(x)$ is the Caputo fractional derivative of order α, L is a linear operator which might include other fractional derivative operators and N is the nonlinear operator which might include other fractional derivative of order less than α, B is boundary operator, $f(t)$ is a known analytic function, Γ is the boundary of domain Ω.

In view of the homotopy perturbation method, we construct the following homotopy

$$(1-p)D^{\alpha}u(t,p)+p\left[D^{\alpha}u(t,p)+Lu(x,t)+Nu(x,t)-f(t)\right]=0 \qquad (3.4)$$

or

$$D^{\alpha}u(t,p)+p\left[Lu(x,t)+Nu(x,t)-f(t)\right]=0 \qquad (3.5)$$

Where $p\in[0,1]$ is an embedding parameter. If $p=0$ equations (3.4) and (3.5) become:

$$D^{\alpha}u=0 \qquad (3.6)$$

and when $p=1$ the equations (3.4)and (3.5) turn out to the fractional partial differential equation (1.1).
The basic assumption is that the solution of Eq.(3.4) or Eq. (3.5) can be written as a power series in p

$$u=u_0+pu_1+p^2u_2+p^3u_3+\cdots\cdots \qquad (3.7)$$.

Substituting (3.7) in (3.5) and equating the terms with having identical power of p, we obtain the following series equations:

$$\left.\begin{array}{l}
p^0: D^{\alpha}u_0 \\
p^1: D^{\alpha}u_1=-Lu_0(t)-N_1u_0(t)+f(t) \\
p^2: D^{\alpha}u_2=-Lu_1(t)-N_2(u_0,u_1) \\
p^3: D^{\alpha}u_3=-Lu_2(t)-N_3(u_0,u_1,u_2) \\
\quad\vdots
\end{array}\right\}$$

Applying the operator J^{α} on both sides of the equations, with considering the initial conditions by using (3.3) the first terms of HPM solution can be given b:

$$\left.\begin{array}{l}
u_0=\sum_{k=0}^{n-1}C_k\dfrac{t^k}{k!} \\
u_1=-J^{\alpha}\left[Lu_0(t)\right]-J^{\alpha}\left[N_1u_0(t)\right]+J^{\alpha}\left[f(t)\right] \\
u_2=-J^{\alpha}\left[Lu_1(t)\right]-J^{\alpha}\left[N_2(u_0,u_1)\right] \\
u_3=-J^{\alpha}\left[Lu_2(t)\right]-J^{\alpha}\left[N_3(u_0,u_1,u_2)\right] \\
\quad\vdots
\end{array}\right\}$$

4. Modified Homotopy Perturbation Method

In [31], Odibat and Momani modified the homotopy perturbation method to solve nonlinear differential equations of fractional order. This modified reduces the nonlinear fractional differential equations to set of linear ordinary differential equations. To illustrate the basic idea of the modification, we consider the following nonlinear fractional differential equation (3.1).

In view of the homotopy technique , we can construct the following homotopy:

$$u^{(m)}+L(u)-f(t)=p\left[u^{(m)}-N(u)-D^{\alpha}u\right] \qquad (4.1)$$

or

$$u^{(m)}-f_1(t)=p\left[u^{(m)}-L(u)-N(u)-D^{\alpha}u+f_2(t)\right],\quad p\in[0,1] \qquad (4.2)$$

Where $f_1(t)$ be assigned to the zeroth component u_0, and $f_2(t)$ be combined with u_1 $\quad i.e\ f(x,t)=f_1(t)+f_2(t)$

In the case of $p=0$ in Eq.(4.2) we get:

$$u^{(m)}=f_1(t) \qquad (4.3)$$

and in the case $p=1$, the Eq. (4.2) turn out to the original fractional differential equating (1.1).

Substituting Eq. (3.7) in to (4.2) and equating the terms with identical powers of p , we obtain the following equations:

$$\left.\begin{array}{ll}
p^0:\dfrac{\partial^m u_0}{\partial t^m}=f_1(x,t), & \dfrac{\partial^k}{\partial t^k}u_0(x,0)=f_k(x),\ k=0,1,2,\ldots \\
p^1:\dfrac{\partial^m u_1}{\partial t^m}=\dfrac{\partial^m u_0}{\partial t^m}-L[x]u_0-N[x]u_0-\dfrac{\partial^\alpha u_0}{\partial t^\alpha}+f_2(x,t), & \dfrac{\partial^k}{\partial t^k}u_1(x,0)=0,\ k=0,1,2,\ldots \\
p^2:\dfrac{\partial^m u_2}{\partial t^m}=\dfrac{\partial^m u_1}{\partial t^m}-L[x]u_1-N[x]u_1-\dfrac{\partial^\alpha u_1}{\partial t^\alpha}, & \dfrac{\partial^k}{\partial t^k}u_2(x,0)=0,\ k=0,1,2,\ldots \\
p^2:\dfrac{\partial^m u_3}{\partial t^m}=\dfrac{\partial^m u_2}{\partial t^m}-L[x]u_2-N[x]u_2-\dfrac{\partial^\alpha u_2}{\partial t^\alpha}, & \dfrac{\partial^k}{\partial t^k}u_3(x,0)=0,\ k=0,1,2,\ldots \\
\vdots &
\end{array}\right\} \qquad (4.4)$$

Not that the equation (4.4) can be solved for u_0, u_1, u_2 by applying J^{α} and by some computations analysis yields to the solution of equation (1.1).

5. Applications

To incorporate our discussion above, five special cases of fractional evolution equation (1.1) will be studied

Example 5.1

In this example we consider the following one dimensional factional partial differential equation of the form

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{1}{2} x^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < \alpha \le 1, \qquad (5.1)$$

Subject to the boundary conditions

$$u(0,t) = 0, \qquad u(1,t) = e^t \qquad (5.2)$$

And initial condition

$$u(x,0) = x^2 \qquad (5.3)$$

The exact solution for the special case $\alpha = 1$ is given by:

$$u(x,t) = x^2 e^t \qquad (5.4)$$

According to homotopy (4.2), substituting the identical condition (5.3) in to (4.4), we get the following set of partial differential equations

$$\left.\begin{array}{ll}
\frac{\partial u_0}{\partial t} = 0, & u_0(x,0) = x^2 \\
\frac{\partial u_1}{\partial t} = \frac{\partial u_0}{\partial t} - \frac{1}{2} x^2 \frac{\partial^2 u_0}{\partial x^2} - \frac{\partial^{\alpha} u_0}{\partial t^{\alpha}}, & u_1(x,0) = 0 \\
\frac{\partial u_2}{\partial t} = \frac{\partial u_1}{\partial t} - \frac{1}{2} x^2 \frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^{\alpha} u_1}{\partial t^{\alpha}}, & u_2(x,0) = 0, \\
\frac{\partial u_3}{\partial t} = \frac{\partial u_2}{\partial t} - \frac{1}{2} x^2 \frac{\partial^2 u_2}{\partial x^2} - \frac{\partial^{\alpha} u_2}{\partial t^{\alpha}}, & u_2(x,0) = 0, \\
\vdots &
\end{array}\right\} \qquad (5.5)$$

Table 1
Approximate solution for some values of α using (5.6)

t/x	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	Exact solution
t=0.1 0.2 0.4 0.6 0.8	0.086828 0.347316 0.781460 1.389261	0.059224 0.236898 0.533021 0.947592	0.048779 0.195117 0.439014 0.780469	0.044206 0.176826 0.397861 0.707306	0.044206 0.176827 0.397861 0.707309
t=0.2 0.2 0.4 0.6 0.8	0.102713 0.410853 0.924419 1.64341	0.070876 0.283506 0.637888 1.134021	0.224509 0.056127 0.505146 0.898038	0.048853 0.195413 0.439681 0.781653	0.0488561 0.195424 0.439704 0.7816977
t=0.3 0.2 0.4 0.6 0.8	0.115024 0.460096 1.035221 1.840391	0.081665 0.326663 0.734993 1.306651	0.063631 0.254527 0.572686 1.018111	0.053981 0.215921 0.485821 0.863682	0.053994 0.2159774 0.485949 0.8639096
t=0.4 0.2 0.4 0.6 0.8	0.125532 0.502130 1.129791 2.000852	0.092158 0.368633 0.829424 1.474531	0.071499 0.285998 0.643496 1.143991	0.059627 0.238507 0.536641 0.954027	0.0596729 0.238691 0.5370568 0.9547678
t=0.5 0.2 0.4 0.6 0.8	0.134903 0.539613 1.214131 2.156452	0.1025540 0.410215 0.922985 1.640061	0.079815 0.319263 0.718342 1.277051	0.065833 0.263333 0.592522 1.053333	0.0659488 0.2637954 0.593539 1.0551816

Consequently, solving the above equations for u_0, u_1, u_2 the first few components of homotopy perturbation solution for equation (5.1) are derived as follows:

$$u_1(x,t) = x^2 \frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$u_2(x,t) = x^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

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Hence the HPM series is

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots$$

$$=x^2\left[1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots\right] \qquad (5.6)$$

Which is exactly the same fourth-order approximate solution obtained in [23] using homotopy analysis method. For especial the case when $\alpha=1$ in (5.6) the exact solution is reached

$$u(x,t)=x^2e^t \qquad (5.7)$$

Table (1) show the 4-term approximate solution for (5.1) for different values of α.

Example 5.2

We consider the following two dimensional factional partial differential equation of the form

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2} \qquad 0<x,y<2\pi, 0<\alpha\le 1, t>0 \qquad (5.8)$$

Subject to the boundary conditions

$$\left.\begin{aligned} u(0,y,t)=0, \qquad & u(2\pi,y,t)=0 \\ u(x,0,t)=0, \qquad & u(x,2\pi,t)=0 \end{aligned}\right\} \qquad (5.9)$$

And initial condition

$$u(x,y,0)=\sin x\,\sin y \qquad (5.10)$$

The exact solution for the special case $\alpha=1$ is given by:

$$u(x,y,t)=e^{-2t}\sin x\,\sin y \qquad (5.11)$$

According to homotopy (4.2), substituting the identical condition (5.10) in to (4.4), we get the following set of partial differential equations

$$\left.\begin{aligned}
&\frac{\partial u_0}{\partial t}=0, && u_0(x,y,0)=\sin x\,\sin y\\
&\frac{\partial u_1}{\partial t}=\frac{\partial u_0}{\partial t}-\frac{\partial^2 u_0}{\partial x^2}-\frac{\partial^2 u_0}{\partial y^2}-\frac{\partial^\alpha u_0}{\partial t^\alpha}, && u_1(x,y,0)=0\\
&\frac{\partial u_2}{\partial t}=\frac{\partial u_1}{\partial t}-\frac{\partial^2 u_1}{\partial x^2}-\frac{\partial^2 u_1}{\partial y^2}-\frac{\partial^\alpha u_1}{\partial t^\alpha}, && u_2(x,y,0)=0,\\
&\frac{\partial u_3}{\partial t}=\frac{\partial u_2}{\partial t}-\frac{\partial^2 u_2}{\partial x^2}-\frac{\partial^2 u_2}{\partial y^2}-\frac{\partial^\alpha u_2}{\partial t^\alpha}, && u_3(x,y,0)=0,\\
&\quad\vdots
\end{aligned}\right\}\qquad(5.12)$$

Consequently, solving the above equations for u_0, u_1, u_2 the first few components of homotopy perturbation solution for equation (5.8) are derived as follows:

$$u_0(x,t)=\sin x\,\sin y\,,$$

$$u_1(x,t)=-2\sin x\,\sin y\,\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2(x,t)=4\sin x\,\sin y\,\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$\vdots$$

Hence the HPM series is

$$\begin{aligned}u(x,t)&=u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots\\
&=\sin x\,\sin y\left[1-2\frac{t^\alpha}{\Gamma(\alpha+1)}+4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-8\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}-\cdots\right]\end{aligned}\qquad(5.13)$$

Which is exactly the same fourth-order approximate solution obtained in [24] using homotopy analysis method. For especial the case when $\alpha=1$ in (5.13) the exact solution is reached

$$u(x,t)=e^{-2t}\sin x\,\sin y \qquad (5.14)$$

Table (2) show the 3-term approximate solution for (5.8) for different values of α.

Example 5.3

We consider the following three dimensional factional partial differential equation of the form

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = x^4 y^4 z^4 + \frac{1}{36}\left[x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + z^2 \frac{\partial^2 u}{\partial z^2}\right] \qquad 0 < x, y, z < 1, 0 < \alpha \leq 1, t > 0 \qquad (5$$

Subject to the boundary conditions

$$\left.\begin{aligned} u(0,y,z,t) &= 0, \qquad u(1,y,z,t) = y^4 z^4 \left(e^t - 1\right), \\ u(x,0,z,t) &= 0, \qquad u(x,1,z,t) = x^4 z^4 \left(e^t - 1\right), \\ u(x,y,0,t) &= 0, \qquad u(x,y,1,t) = x^4 y^4 \left(e^t - 1\right) \end{aligned}\right\} \qquad (5.16)$$

And initial condition

$$u(x,y,z,0) = 0 \qquad (5.17)$$

The exact solution for the special case $\alpha = 1$ is given by:

$$u(x,y,z,t) = x^4 y^4 z^4 \left(e^t - 1\right) \qquad (5.18)$$

According to homotopy (4.2), substituting the identical condition (5.17) in to (4.4), we get the following set of partial differential equations

Table 2

Approximate solution for some values of α using (5.13)

t/x	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	Exact solution
t=0.1	0.086828	0.059224	0.048779	0.044206	0.044206
0.2	0.347316	0.236898	0.195117	0.176826	0.176827
0.4	0.781460	0.533021	0.439014	0.397861	0.397861
0.6	1.389261	0.947592	0.780469	0.707306	0.707309
0.8					
t=0.2	0.102713	0.070876	0.224509	0.048853	0.0488561
0.2	0.410853	0.283506	0.056127	0.195413	0.195424
0.4	0.924419	0.637888	0.505146	0.439681	0.439704
0.6	1.64341		0.898038	0.781653	0.7816977
0.8		1.134021			
t=0.3	0.115024	0.081665		0.053981	0.053994
0.2	0.460096	0.326663		0.215921	0.2159774

0.4 0.6 0.8	1.035221 1.840391	0.734993 1.306651		0.485821 0.863682	0.485949 0.8639096
t=0.4 0.2 0.4 0.6 0.8	0.125532 0.502130 1.129791 2.000852	0.092158 0.368633 0.829424 1.474531	0.071499 0.285998 0.643496 1.143991	0.059627 0.238507 0.536641 0.954027	0.0596729 0.238691 0.5370568 0.9547678
t=0.5 0.2 0.4 0.6 0.8	0.134903 0.539613 1.214131 2.156452	0.1025540 0.410215 0.922985 1.640061	0.079815 0.319263 0.718342 1.277051	0.065833 0.263333 0.592522 1.053333	0.0659488 0.2637954 0.593539 1.0551816

Consequently, solving the above equations for u_0, u_1, u_2 the first few components of homotopy perturbation solution for equation (5.15) are derived as follows:

$$u_0(x,y,z,t) = 0,$$

$$u_1(x,y,z,t) = x^4 y^4 z^4 \frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$u_2(x,y,z,t) = -x^4 y^4 z^4 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$\vdots$$

Hence the HPM series is

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$

$$= x^4 y^4 z^4 \left[\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \cdots \right] \qquad (5.20)$$

Which is exactly the same fourth-order approximate solution obtained in [23] using homotopy analysis method. For especial the case when $\alpha = 1$ in (5.20) the exact solution is reached

$$u(x,y,z,t) = x^4 y^4 z^4 \left(e^t - 1\right) \qquad (5.21)$$

Table (3) show the 3-term approximate solution for (5.15) for different values of α.

Table 3
Approximate solution for some values of α using (5.20)

t/x	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	Exact solution
t=0.1 0.2 0.4 0.6 0.8	$1.87E^{-9}$ $7.67E^{-6}$ $9.95E^{-4}$ $3.14E^{-2}$	$1.15E^{-9}$ $4.\ 71E^{-6}$ $6.11E^{-4}$ $1.93E^{-2}$	$7.04E^{-9}$ $2.88E^{-6}$ $3.74E^{-4}$ $1.18E^{-2}$	$3.90E^{-10}$ $1.60E^{-6}$ $2.07E^{-4}$ $6.54E^{-2}$	$4E^{-10}$ $1.7E^{-6}$ $2.2E^{-4}$ $7.3E^{-2}$
t=0.2 0.2 0.4 0.6 0.8	$2.29E^{-9}$ $9.37E^{-6}$ $1.22E^{-4}$ $3.84E^{-2}$	$1.15E^{-9}$ $6.24E^{-6}$ $8.07E^{-4}$ $2.55E^{-2}$	$1.10E^{-10}$ $4.51E^{-6}$ $5.85E^{-4}$ $1.85E^{-2}$	$7.43E^{-10}$ $3.04E^{-6}$ $3.94E^{-4}$ $1.25E^{-2}$	$9E^{-10}$ $3.71E^{-6}$ $4.8E^{-4}$ $1.52E^{-2}$
t=0.3 0.2 0.4 0.6 0.8	$12.62E^{-9}$ $1.07E^{-5}$ $1.39E^{-3}$ $4.39E^{-2}$	$1.81E^{-9}$ $7.41E^{-6}$ $9.61E^{-4}$ $3.03E^{-2}$	$1.41E^{-9}$ $5.76E^{-6}$ $7.48E^{-4}$ $2.36E^{-2}$	$1.06E^{-9}$ $4.35E^{-6}$ $5.65E^{-4}$ $1.78E^{-2}$	$1.4E^{-9}$ $5.86E^{-6}$ $7.6E^{-4}$ $2.4E^{-2}$
t=0.4 0.2 0.4 0.6 0.8	$2.91E^{-9}$ $1.197E^{-5}$ $1.54E^{-3}$ $4.00E^{-2}$	$2.06E^{-9}$ $8.45E^{-6}$ $1.09E^{-4}$ $3.46E^{-2}$	$1.51E^{-9}$ $6.20E^{-6}$ $8.04E^{-4}$ $2.54E^{-2}$	$1.35E^{-9}$ $5.55E^{-6}$ $7.20E^{-4}$ $2.23E^{-2}$	$2E^{-10}$ $8.2E^{-6}$ $10.7E^{-4}$ $3.3E^{-2}$
t=0.5 0.2 0.4 0.6 0.8	$3.18E^{-9}$ $1.30E^{-5}$ $1.69E^{-3}$ $5.34E^{-2}$	$2.31E^{-9}$ $9.46E^{-6}$ $1.23E^{-3}$ $3.87E^{-2}$	$1.89E^{-9}$ $7.78E^{-6}$ $1.00E^{-3}$ $3.18E^{-2}$	$1.62E^{-9}$ $6.64E^{-6}$ $8.62E^{-4}$ $2.72E^{-2}$	$2.16E^{-9}$ $10.81E^{-6}$ $1.4E^{-4}$ $4.4E^{-2}$

Example 5.4

We consider the following three dimensional inhomogeneous factional partial differential equation of the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{12}\left[x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2}\right] \qquad 0 < x, y < 1, 0 < \alpha \le 2, t > 0 \qquad (5.22)$$

Subject to the boundary conditions

$$\left.\begin{aligned} &u(0,y,t)=0, \qquad u(1,y,t)=4\cosh t, \\ &u(x,0,t)=0, \qquad u(x,1,t)=4\sinh t, \end{aligned}\right\} \qquad (5.23)$$

And initial condition

$$u(x,y,0)=x^4, \qquad u_t(x,y,0)=y^4 \qquad (5.24)$$

The exact solution for the special case $\alpha=1$ is given by :

$$u(x,y,t)=x^4\cosh t+y^4\sinh t \qquad (5.25)$$

According to homotopy (4.2), substituting the identical condition (5.24) in to (4.4), we get the following set of partial differential equations

$$\left.\begin{aligned} &\frac{\partial^2 u_0}{\partial t^2}=0 \qquad u_0(x,y,0)=x^4, \qquad u_t(x,y,0)=y^4 \\ &\frac{\partial^2 u_1}{\partial t^2}=\frac{\partial^2 u_0}{\partial t^2}-\frac{1}{12}\left[x^2\frac{\partial^2 u_0}{\partial x^2}+y^2\frac{\partial^2 u_0}{\partial y^2}\right]-\frac{\partial^\alpha u_1}{\partial t^\alpha}, \qquad \frac{\partial}{\partial t}u_1(x,y,0)=0 \\ &\frac{\partial^2 u_2}{\partial t^2}=\frac{\partial^2 u_1}{\partial t^2}-\frac{1}{12}\left[x^2\frac{\partial^2 u_1}{\partial x^2}+y^2\frac{\partial^2 u_1}{\partial y^2}\right]-\frac{\partial^\alpha u_2}{\partial t^\alpha}, \qquad \frac{\partial}{\partial t}u_2(x,y,0)=0 \\ &\quad \mathrm{M} \end{aligned}\right\} \qquad (5.26)$$

Solving these equations for u_0, u_1, u_2, we get

$$u_0(x,y,t)=x^4+y^4 t$$

$$u_1(x,y,t)=x^4\frac{t^\alpha}{\Gamma(\alpha+1)}+y^4\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2(x,y,t)=x^4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+y^4\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)},$$

$$u_3(x,y,t)=x^4\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+y^4\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)},$$

M

Hence the HPM series is

$$\left.\begin{aligned}
u(x,t) &= u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots \\
&= x^4+y^4t+x^4\frac{t^{\alpha}}{\Gamma(\alpha+1)}+y^4\frac{t^{\alpha}}{\Gamma(\alpha+2)}+x^4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+y^4\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \\
&= x^4\left[1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots\right]+y^4\left[t+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\cdots\right]
\end{aligned}\right\}(5.27)$$

Which is exactly the same fourth-order approximate solution obtained in [23] using homotopy analysis method. For especial the case when $\alpha=2$ in (5.27) the exact solution is reached

$$u(x,y,t)=x^4\cosh t+y^4\sinh t \qquad (5.28)$$

Table (4) show the 3-term approximate solution for (5.22) for different values of α.

Table 4

Approximate solution for some values of α using (5.26)

t	x	y	α = 1.25	α = 1.5	α = 1.75	α = 2	Exact solution
t=0.2	0.01	0.25	0.0008737	0.0008025	0.0007918	0.0007864	0.0007864
		0.75	0.0667005	0.0650056	0.0641413	0.0637039	0.0637039
	0.05	0.25	0.0008304	0.0008092	0.0007983	0.0007928	0.0007928
		0.75	0.0667075	0.0650122	0.0641478	0.0637103	0.0637103
t=0.5	0.01	0.25	0.0023607	0.0021739	.0020877	0.0020355	0.0020355
		0.75	0.1868849	0.1758817	0.1691078	0.1648778	0.1648778
	0.05	0.25	0.0023157	0.0021794	0.0020952	0.0020425	0.0020425
		0.75	0.1868580	0.1758898	0.1691153	0.1648848	0.1648848
t=0.8	0.0	0.2	0.004223	0.003868	0.003631	.0034691	.003469

	1 0.0 5	5 4 0.7 0.342098 5 4 0.2 0.004235 5 1 0.7 0.342110 5 1	4 0.313346 8 0.003878 6 0.313335 6	3 0.294135 6 0.003640 4 0.294144 1	0.281002 1 0.003477 5 0.281010 5	1 0.281002 3 0.003477 5 0.281010 6

Example 5.5

We consider the following three dimensional inhomogeneous factional partial differential equation of the form

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\left(x^2+y^2+z^2\right)+\frac{1}{2}\left[x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}\right] \qquad 0<x,y,z<1, 0<\alpha\le 2, t>0 \qquad (5.29)$$

Subject to the boundary conditions

$$\left.\begin{aligned} &u(0,y,z,t)=y^2\left(e^t-1\right)+z^2\left(e^{-t}-1\right),\ u(1,y,z,t)=\left(1+y^2\right)\left(e^t-1\right)+z^2\left(e^{-t}-1\right),\\ &u(x,0,z,t)=x^2\left(e^t-1\right)+z^2\left(e^{-t}-1\right),\ u(x,1,z,t)=\left(1+x^2\right)\left(e^t-1\right)+z^2\left(e^{-t}-1\right),\\ &u(x,y,0,t)=\left(x^2\right)\left(e^t-1\right)+y^2\left(e^{-t}-1\right),\ u(x,y,1,t)=\left(1+x^2\right)\left(e^t-1\right)+y^2\left(e^{-t}-1\right),\end{aligned}\right\} \qquad (5.30)$$

and initial condition

$$u(x,y,z,0)=0, \qquad u_t(x,y,z,0)=x^2+y^2-z^2 \qquad (5.31)$$

The exact solution for the special case $\alpha=2$ is given by:

$$u(x,y,z,t)=-\left(x^2+y^2+z^2\right)+\left(x^2+y^2\right)e^{-t}+z^2e^{-t} \qquad (5.32)$$

According to homotopy (4.2), substituting the identical condition (5.31) in to (4.4), we get the following set of partial differential equations

$$\left.\begin{aligned} &\frac{\partial^2 u_0}{\partial t^2}=-\left(x^2+y^2-z^2\right) \qquad u_0(x,y,z,0)=0, \qquad u_t(x,z,y,0)=-\left(x^2+y^2+z^2\right)\\ &\frac{\partial^2 u_1}{\partial t^2}=\frac{\partial^2 u_0}{\partial t^2}-\frac{1}{2}\left[x^2\frac{\partial^2 u_0}{\partial x^2}+y^2\frac{\partial^2 u_0}{\partial y^2}+z^2\frac{\partial^2 u_0}{\partial z^2}\right]-\frac{\partial^\alpha u_1}{\partial t^\alpha}-\left(x^2+y^2+z^2\right),\ \frac{\partial}{\partial t}u_2(x,y,0)=0\\ &\frac{\partial^2 u_2}{\partial t^2}=\frac{\partial^2 u_1}{\partial t^2}-\frac{1}{2}\left[x^2\frac{\partial^2 u_1}{\partial x^2}+y^2\frac{\partial^2 u_1}{\partial y^2}+z^2\frac{\partial^2 u_1}{\partial z^2}\right]-\frac{\partial^\alpha u_2}{\partial t^\alpha}, \qquad \frac{\partial}{\partial t}u_2(x,y,z,0)=0\\ &\quad \mathrm{M}\end{aligned}\right\} \qquad (5.33)$$

Solving these equations for u_0,u_1,u_2, we get

Hence the HPM series is

$$
\left.\begin{aligned}
u(x,t) &= u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots \\
&= \left(x^2+y^2+z^2\right)t+\left(x^2+y^2+z^2\right)\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots\right] \\
&\quad +\left(x^2+y^2-z^2\right)\left[\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}+\cdots\right] \\
&= -\left(x^2+y^2+z^2\right)+\left(x^2+y^2\right)\left[1+t+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots\right] \\
&\quad +z^2\left[1-t+\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\cdots\right]
\end{aligned}\right\} \quad (5.34)
$$

Which is exactly the same fourth-order approximate solution obtained in [23] using homotopy analysis method. For especial the case when $\alpha = 2$ in (5.34) the exact solution is reached

$$u(x,y,z,t) = -\left(x^2+y^2+z^2\right)+\left(x^2+y^2\right)e^{t}+z^2e^{-t} \qquad (5.35)$$

Table (5) show the 3-term approximate solution for (5.29) for different values of α.

Table 5
Approximate solution for some values of α using (5.26)

t	x	y	z	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2$	Exact solution
t=0.2	0.025	0.04	0.04	0.0005801	0.0037877	0.0002640	0.0001997	0.0002026
			0.08	0.0001607	-0.000278	-0.000528	-0.000670	-0.000667
		0.08	0.04	0.0021333	0.0016681	0.0014040	0.0012559	0.0012653
			0.08	0.0017139	0.0010114	0.0006110	0.0003858	0.0003952
t=0.5	0.025	0.04	0.04	0.0018012	0.0013220	0.0009380	0.0007668	0.0008139
			0.08	0.0010055	0.0003088	-0.0006478	-0.001121	-0.001074
		0.08	0.04	0.0062593	0.0051027	0.00431822	0.0037794	0.0039276
			0.08	0.0054635	0.0038115	0.00267659	0.0018907	0.0022039
t=0.8	0.025	0.04	0.04	0.0033283	0.0027508	0.00205665	0.0016491	0.0018458
		0.08	0.08	0.0021976	0.0008425	-0.0001958	-0.000995	-0.000797
			0.04	0.1132574	0.0094648	0.00811327	0.0071088	0.0077283
			0.08	0.0101892	0.0077094	0.00648113	0.0044640	0.0050851

6. Conclusion

In this paper, Modified of homotopy perturbation method (MHPM) has been applied to solving the fractional partial differential equations with variable coefficients. The proposed method was clearly very efficient and powerful method in finding the solutions of proposed equations. For illustration purpose, we consider five different examples. The implementation of the modified homotopy perturbation method reduces the fractional partial differential equation to set of simple differential equations, which are easy to solve.

The study shows that the method requires less computational work, where in the special case $\alpha = 1$ and $\alpha = 2$, the general solution reduces to the partial differential equations. On the other hand, the agreement between the approximate and the exact solution in all examples conclude that the efficiency of the method and related phenomena give the method much wider applicability.

7. References

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